ON THE COMPARISON BETWEEN PICARD'S ITERATION METHOD AND ADOMIAN DECOMPOSITION METHOD IN SOLVING LINEAR & NON-LINEAR DIFFERENTIAL EQUATIONS

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BY

YOHANNES TESFAYE

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Yohannes Tesfaye

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PREFACE

This seminar paper is concerned, on the comparison between Picard's iteration method and Adomian decomp0ostion method in solving linear and non-linear differential equations. It consists of six units. The first unit deals with the introduction of the paper mainly the background, statement of the problem, objective and the methodology. The second unit deals with the main body of the seminar. That is on the comparison between Picard's iteration method and Adomian decomposition method in solving linear and non-linear differential equations. Several numerical methods discussed in detail. The third some real life applications of linear and non-linear partial differential equations are discussed using Picard's iteration and Adomian decomposition method. The fourth unit contains summary of the seminar paper.

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1. INTRODUCTION

1.1. Background of the study

Most of the problems that arise in the real world are described by differential equations. However, many differential equations cannot be solved by existing standard methods. In such equations, since it is difficult to find an exact solution it is sufficient to obtain an approximate solution. Among the approximation method used for solving differential equations, are the Picard iteration method and the A domian decomposition method.

Large classes of linear differential equations can be solved by A domain decomposition method (ADM) and Picard iteration method. The ADM was first compared with the Picard method by Rachand Bellomo and Sarafyan on a number of examples. Golberg showed that A domain approach to linear differential equations was equivalent to the classical method of successive approximations (Picard method).

Picard iteration method (PIM) named after a French mathematician Charles Emile Picard (1856-1941) is a successive approximation method which is used in giving an approximate solution to an initial-value problem. Many scholars have worked on the Picard iteration method over the years .Some classes of differential equations which include initial value ,boundary value, and Eigen value problems arising in physics and engineering are shown to be solvable by a method which in volves the combination of an explicit or direct integration approach with a suitable modified Picard iterative methods. Examples involving partial and ordinary differential equations were presented by Robin(2010),the solution obtained are in the power series form and the method can be easily implemented on the computer.

The Adomian decomposition method is an approximation method used for solving non-linear differential equation.This method involves the decomposition of a non-linear equations into solutions in a series of functions. These series of functions are obtained in a recursive procedure from a polynomial called Adomian polynomials which are initiated from the expansion of an analytic function into a power series function.ADM is most useful in non-linear differential equations and their accuracy is determined by the number of the partial solutions used.

Studies have been conducted on the ADM over the years by different scholars. A modified approach to the Adomian polynomials which converges a little faster than the original Adomian polynomials and are favourable for computer generation was introduced by Adomian. These modifications have been used to solve singular and non-singular differential equations effectively. A study for an effective modification of the ADM for solving second-order singular initial value problems was done and it was discovered that with few iterations used, the ADM is simple, easy to use and produces reliable results. The proper use of the ADM has made it possible to obtain an analytic solution of a singular initial-value problem when it is homogeneous or inhomogeneous. Some modifications of the ADM were presented for solving initial-value problems and its effectiveness was verified. The structure of a new successive initial solution, which can give a more accurate solution when treating an initial boundary-value problem by mixed initial and boundary conditions to obtain a new initial solution at every iteration steps of the ADM, was discovered. A simple program was introduced which solves the initial-value problem, and its simplicity and efficiency was presented.

A simple method to determine the rate of convergence of the ADM was investigated and it was discovered that the modified ADM converges faster than the standard ADM when comparing the two methods The appearance of noise terms in the decomposition method was investigated and it was discovered that the noise terms plays a vital role in increasing the rate of convergence of the solution of quadratic integral solutions as presented by A new proof used for the evaluation of the numbers and also for demonstrating a new result for Bell polynomials was established and it also proved a new norm bound for A domian polynomials A new and strong method for solving non-linear equations of different kinds was implemented The method gave new and sufficient conditions for obtaining convergence of the decomposition series, and also such method can be used to prove the convergence of a regularisation method which can be applied to integral equations of the first kind.

The convergence of ADM when applied to wave, time-dependent heat and beam equations for both forward and backward time evolutions converges faster for forward problems than backward problems the convergence of the ADM by using the Cauchy Kovalevskaya theorem for differential equations and he obtained a new result on the convergence rate of ADM. He also compared this method and Picard iteration method and he discovered that the ADM converges faster than the Picard iteration method.

In this study, the A domain decomposition method (ADM) and Picard iteration method (PIM) for linear and non-linear differential equations will be compared in terms of the convergence, The PIM and ADM will be analyzed with its application to initial-value problems, also the Picard existence and uniqueness and the rate and order of convergence of the ADM will be proved. Some non-linear differential equations will be considered and numerical approximations will be done, the convergence, efficiency and accuracy of the methods will be verified.

1.2 statement of the problem

The purpose of this seminar paper is to review some recently developed PIM and ADM methods for solving linear and non-linear differential equations. This seminar focuses on the comparison between the PIM and ADM methods for solving linear and non-linear differential equations. The PIM and ADM methods will be analyzed with its applications to initial-value problems. In the case of PIM initiating the existence and uniqu existence and uniqueness of a solution to differential equation of the form $y' = f(x, y)$, and how to check the performance of the method (In terms of accuracy, efficiency and convergence).

The seminar will also discuss on the problem to study which method computationally efficient and less accurate for solving linear and non-linear differential equations. for considering the convergence rates, when the two analytical series based methods are converges (diverges) in a given differential equations and where are the two methods combined also studied in this seminar.

1.3 Objectives of the study

The main purpose of this seminar is to present are view of the two analytical series based method to solve both linear and non-linear differential equations which is intended to alive the following objective.

- To discuss the PIM and ADM solution method
- To solve linear differential equation using the PIM method
- To demonstrate examples using PIM and ADM solution
- To solve non-linear differential equation using the ADM method
- To determine the convergence and error estimating of the PIM and ADM method's.

1.4 Methodologies

Any academic activity such as research or project works has its own methodology. So this study work has the following methodology:

- ✓ The seminars involved collecting information from some books available in libraries and also search the internet regularly. All the information obtained was recorded.
- ✓ The collected information (definitions, theorems, methods) examined in detail.
- ✓ The facts and concepts related to the specified topic were noted.
- ✓ The investigators was collect seminar articles and prepare reports by making a frequent contact with the advisor throughout the seminar work

2. PICARD ITERATION METHOD (PIM)

The Picard iteration method is an iteration method which is used to provide an existence solution to initial value problems. In this chapter, the analysis of the Picard iteration method was showed and the detailed proof of the existence and uniqueness theorem was given.

2.1 Analysis of the picard iteration method

In this section, the Picard iteration method will be analyzed.

Consider an initial value problem:

$$y'(x) = f(x, y), \quad y(x_o) = y_o \tag{2.1.1}$$

Equation (2.1.1) can be written in an integral form:

$$\int_{x_o}^{x} y'(x)dx = \int_{x_o}^{x} f(x, y(x))dx$$

$$y(x) - y(x_o) = \int_{x_o}^{x} f(x, y(x))dx \tag{2.1.2}$$

Substituting the initial condition $y(x_0) = y_0$ in to equation (2.1.2) we have:

$$y(x) - y_o = \int_{x_o}^{x} f(x, y(x))dx \tag{2.1.3}$$

Changing the variable of integration to t instead of x in (2.1.3) gives,

$$y(x) = y_o + \int_{x_o}^{x} f(t, y(t))dt \tag{2.1.4}$$

Equation (2.1.4) is used to generate successive approximation of solution to initial-value problems which is of the form:

$$y_n(x) = y_o + \int_{x_o}^{x} f(t, y_{n-1}(t))dt \quad n = 1,2,3 \tag{2.1.5}$$

Hence the successive approximations are of the form:

$y_o(x) = y_o,$

$y_1(x) = y_o + \int_{x_o}^{x} f(t, y_o(t),$

$y_2(x) = y_o + \int_{x_o}^{x} f(t, y_1(t)$

$\vdots$

$y_{n+1}(x) = y_o + \int_{x_o}^{x} f(t, y_n(t))dt,$...(2.1.6)

2.2. Picard's Existence and Uniqueness Theorem (Cauchy Lipschitz Theorem)

In this section, we prove the Picard's existence and uniqueness theorem as presented

Theorem, Let $\mathbf{f}$(x, y) be a continuous function of (x, y) plane in a region D. Let M be a Constant such that

$$| \mathbf{f}(x, y) | < M \quad \text{in} \quad D. \qquad (2.2.1)$$

Also, let $\mathbf{f}$(x, y) in D satisfy the Lipschitz condition:

$$|f(x, y_1) - f(x, y_2)| \leq A|y_1 - y_2|, \qquad (2.2.2)$$

Where A is independent of x, y_1, y_2.

Suppose there exist a rectangle R defined by

$$|x - x_0| \leq h, \quad | y - y_0| \leq k, \qquad (2.2.3)$$

such that $Mh < k$ where $R \subseteq D$.

Then for $|x - x_0| \leq h$, y' $= \mathbf{f}$(x, y) has a unique solution y = y(x) for which $y(x_0) = y_0$.

Proof. Assume that x is defined such that $|x - x_0| \leq h$.

Let's also define the series of functions:

$$y_o(x) = y_o,$$

$$y_1(x) = y_o + \int_{x_o}^{x} f(x, y_o(x))dx$$

$$y_2(x) = y_o + \int_{x_o}^{x} f(x, y_1(x))dx$$

$$y_3(x) = y_o + \int_{x_o}^{x} f(x, y_2(x))dx$$

$$\vdots$$

$$y_{n-1}(x) = y_o + \int_{x_o}^{x} f(x, y_{n-2}(x))dx$$

$$y_n(x) = y_o + \int_{x_o}^{x} f(x, y_{n-1}(x))dx \qquad (2.2.4)$$

We will then consider the proof in five main steps:

First step: To show that for $x_0 - h \le x \le x_0 + h$, there exists a curve $y = y_n(x)$ which lies in the rectangle R i.e. $\exists \; y_0 - k \le y \le y_0 + k$.

Now, suppose $y = y_{n-1}(x)$ lies in R so that $f(x, y_{n-1}(x))$ is defined, continuous and satisfies

$|f(x, y_{n-1}(x)| \le M$ on the interval $x_0 - h \le x \le x_0 + h$.

Then from equation (2.1.4),

$$\begin{aligned} |y_n(x) - y_o(x)| &= \left|\int_{x_o}^{x} f(x, y_{n-1}(x)dx\right|, \\ &\le M|x - x_o|, \\ &\le Mh < k, \end{aligned}$$

We have clearly shown that $y_n(x)$ lies in R, and hence $f(x, y_n(x))$ is defined and continuous on the interval $x_0 - h \le x \le x_0 + h$.

Second step: To show that by induction $y = y_n(x)$ lies in R so that:

$$|y_n(x) - y_{n-1}(x)| \le \frac{MA^{n-1}}{n!}|x - x_o|^n. \qquad (2.2.5)$$

For $n = 1$,

$$|y_1(x) - y_o(x)| \le M|x - x_o|$$

$\le Mh < k$,

For n= n-1:

let $$\left|y_{n-1(x)} - y_{n-2}(x)\right| \le \frac{MA^{n-1}}{n-1!}\left|x - x_o\right|^{n-1} \quad (2.2.6)$$

And from equation (2.2.4),we have that

$$\left|y_n(x) - y_{n-1}(x)\right| = \left|\int_{x_o}^{x} (f(x, y_{n-2}(x) - f(x, y_{n-1}(x))dx\right|, \quad (2.2.7)$$

$$\left|y_n(x) - y_{n-1}(x)\right| \le \int_{x_o}^{x} \left|f(x, y_{n-2}(x) - f(x, y_{n-1}(x)\right|\left|dx\right|.$$

From equation in equation (2.2.2) gives:

$$\left|f(x, y_{n-2}(x)) - f(x, y_{n-1}(x)\right| < A\left|y_{n-1}(x) - y_{n-2}(x)\right|. \quad (2.2.8)$$

From equation (2.2.7) and (2.2.8), we have:

$$\left|y_n(x) - y_{n-1}(x)\right| \le \int_{x_o}^{x} A\left|y_{n-1}(x) - y_{n-2}(x)\right|\left|dx\right|,$$

$$\left|y_n(x) - y_{n-1}(x)\right| \le A\int_{x_o}^{x} \left|y_{n-1}(x) - y_{n-2}(x)\right|\left|dx\right|, \quad (2.2.9)$$

And from equation (2.2.6)

$$\left|y_n(x) - y_{n-1}(x)\right| \le A\int_{x_o}^{x} \frac{MA^{n-2}}{(n-1)!}\left|x - x_o\right|^{n-1}\left|dx\right|,$$

$$\left|y_n(x) - y_{n-1}(x)\right| \le A\frac{MA^{n-2}}{(n-1)!}\int_{x_o}^{x} \left|x - x_o\right|^{n-1}\left|dx\right|,$$

$$\left|y_n(x) - y_{n-1}(x)\right| \le \frac{MA^{n-2}}{(n-1)!}\int_{x_o}^{x} (x - x_o)^{n-1}dx,$$

$$\left|y_n(x) - y_{n-1}(x)\right| \le \frac{MA^{n-1}}{(n-1)!}\frac{(x - x_o)^n}{n}.$$

Thus by induction,

$$\left|y_n(x) - y_{n-1}(x)\right| \le \frac{MA^{n-1}}{(n)!}\left|x - x_o\right|^n \forall_n \in N. \quad (2.2.10)$$

Third step: Now we will show that sequence $y_n(x)$ converges uniformly to a limit

$x_o - h \le x \le x_o + h$ For interval

$$|x - x_o| \le h. \tag{2.2.11}$$

Substitute equation (2.2.11) into (2.2.10),

$$|y_n(x) - y_{n-1}(x)| \le \frac{MA^{n-1}}{(n)!} h^n. \tag{2.2.12}$$

By infinite series and the fact that the difference is bounded, equation 2.2.12 becomes:

$$y_o(x) + (y_1(x) - y_o(x)) + (y_2(x) - y_1(x)) + \ldots + (y_n(x) - y_{n-1}(x)) + \ldots \le y_o + Mh + \frac{1}{2!} Mkh^2 + \ldots$$
$$+ \frac{1}{n!} Mk^{n-1}.$$
$$h^n + \ldots, y_o(x) + (y_1(x) - y_o(x)) + (y_2(x) - y_1(x)) + \ldots + (y_n(x) - y_{n-1}(x)) + \ldots$$
$$\le y_o + \frac{M}{k}(e^{kh} - 1) \quad \text{.......................................}(2.2.13)$$

Consequently, the series in equation (2.2.13) is absolutely convergent for all h. hence by weiersrass m-test, the series converges uniformly for $x_o - h \le x \le x_o + h.$ since the terms of the series are continuous in x, then the sum $\lim_{n \to \infty} y_n(x) = y(x)$ is also continuous.

Fourth step: We will now show that y = y(x) satisfies the differential equation

$$y' = f(x, y).$$

Since $y_n(x)$ converges uniformly to $y(x) in [x_o - h, x_o + h]$ by Lipschitz condition (2.2.2), we then have that

$$|f(x, y(x)) - f(x, y_n(x))| \le A|y(x) - y_n(x)|$$

It follows that $f(x, y_n(x))$ tends uniformly to f(x,y(x)).

If we let $n \to \infty$ in equation (2.2.4), we will have:

$$\lim_{n\to\infty} y_n(x) = y_o + \lim_{n\to\infty}\int_{x_o}^{x} f(x, y_{n-1}(x)dx, \qquad (2.2.14)$$

$$y(x) = y_o + \int_{x_o}^{x} \lim_{n\to\infty} f(x, y_{n-1}(x)dx. \qquad (2.2.15)$$

Since $f(x, y_n(x))$ is a sequence which consists of a continuous function which converges uniformly to $f(x, y(x))$ on the same interval,

hence

$$y(x) = y_o + \int_{x_o}^{x} f(x, y(x)dx. \qquad (2.2.16)$$

The integrand $f(x, y(x))$ is a continuous function of x. Therefore the integral has the derivative $f(x, y)$ and $y(x_0) = y_0$.

Finally, we will show that the solution $y = y(x)$ is unique and $y(x_0) = y_0$. Lets assume $y = Y(x)$ so that,

$$|Y(x) - y(x)| \leq c, \qquad (2.2.17)$$

Where $x_0 - h \leq x \leq x_0 + h$ and $c = 2k$.

From equation (3.1.16), we have;

$$|Y(x) - y(x)| = \left|\int_{x_o}^{x} f(x, Y(x)) - f(x, y(x)dx\right|,$$

$$|Y(x) - y(x)| = A\int_{x_o}^{x} |Y(x) - y(x)||dx|, \qquad (2.2.18)$$

$$|Y(x) - y(x)| \leq Ac|x - x_o|. \qquad (2.2.19)$$

Now substituting equation (2.1.16) into (2.1.18), to have;

$$|Y(x) - y(x)| \leq A^2 c\int_{x_o}^{x} |x - x_o|dx,$$

$$|Y(x) - y(x)| \leq \frac{A^2 c|x - x_o|^2}{2!} \qquad (2.2.20)$$

Again substituting equation (2.2.20) into (2.2.18), gives;

$$|Y(x) - y(x)| \leq \frac{A^3 c}{3!}|x - x_o|^3,$$

And recursively becomes

$|Y(x) - y(x)| \leq \frac{c(Ah)^n}{n!}$, *where*, $|x - x_o| = h$,

But the series $\sum_{k=0}^{n} \frac{c(Ah)^n}{n!}$ converges so that $\lim_{n \to \infty} \frac{c(Ah)^n}{n!} = o.$

Thus $|Y(x) - y(x)| \to 0$ and we conclude that $Y(x) = y(x)$ which shows that $y = y(x)$ is unique

3. ADOMIAN DECOMPOSITION METHOD (ADM)

The Adomian decomposition method (ADM) is an approximation method used for solving non-linear differential equation, (Adomian, 1988, 1992). This method involves the decomposition of non-linear equations into solutions in a series of functions. These series of functions are obtained in a recursive procedure from a polynomial called Adomian polynomials which are initiated from the expansion of an analytic function into a power series function.

3.1. Analysis of the Adomian decomposition method

In this section, we present the derivation of the Adomian decomposition method. Consider the equation

$$y' + f(x, y) = g(x), \quad y(0) = A \ . \tag{3.1.1}$$

Where **f**(x,y) is real, non-linear and continuous. g(x)∈C[0,1] are linear functions, and A is constant. We define the differential operator L as

$$L = \frac{d}{dx},$$

And

$$L_y = \frac{d}{dx}(y) = y'$$

$$L_y = y' \tag{3.1.2}$$

Hence equation (3.1.1) becomes

$$L_y + f(x, y) = g(x), \tag{3.1.3}$$

This can also be written as:

$$L_y = g(x) - f(x, y) \tag{3.1.4}$$

And the inverse operator which is considered as a linear integral operator (n-fold integral) is given as:

$$L^{-}(.) = \int_{0}^{x} (.)dx \qquad (3.1.5)$$

When we find the inverse of L_y in equation (3.1.4), it gives:

$$L^{-1}(L_y) = L^{-1}(g(x)) - L^{-1}(f(x, y)).. \qquad (3.1.6)$$

Substituting L_y from(3.1.2) into (3.1.6) to obtain

$$L^{-1}(y') = L^{-1}(g(x)) - L^{-1}(f(x, y)).. \qquad (3.1.7)$$

Evaluating the left hand side of equation (3.1.7), we obtain:

$$L^{-1}(y') = \int_{x_0}^{x} (y')dx$$

Computing the integral and using the initial value condition given to obtain

$$L^{-1}(y') = \int_{x_0}^{x} (y')dx = y(x) + A \qquad (3.1.8)$$

Where A is a constant of integration

Substituting equation (3.1.8) into (3.1.7),

$$y(x) + A = L^{-1}(g(x)) + L^{-1}(f(x, y)).$$

Hence

$$y(x) = A - L^{-1}(g(x)) + L^{-1}(f(x, y)). \qquad (3.1.9)$$

Adomian decomposition method introduces the solution y(x) in an infinite series form;

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \qquad (3.1.10)$$

The components of $y_n(x)$ are determined recursively.

The Adomian decomposition method also defines the non-linear function by an infinite polynomial series of the form:

$$f(x, y)= \sum_{n=0}^{\infty} A_n \qquad (3.1.11)$$

The A_n 's are called the Adomian polynomials which can be constructed using the general formula for a non-linear function F (u),

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\text{a}^n}\left[F(\sum_{i=0}^{n} \text{a}^i u^i)\right]\right]_{\text{a}=0}, \qquad n=0,1,2,\ldots,$$

Where a is a parameter introduced for convenience.

Constructing the first four iterations for A_n,

$$Ao = F(u_o),$$

$$A_1 = \left[\frac{d}{d\text{a}} F(u_o + u_1 \text{a})\right]_{\text{a}=0} = u_1 F'(u_o),$$

$$A_2 = \frac{1}{2!}\frac{d}{d\text{a}}\left[((u_1 + 2u_2 \text{a})F'(u_o + u_1 \text{a}))\right]_{\text{a}=0} = u^2 F'(u_o) + \frac{1}{2!}u_1^2 F''(u_o).$$

$$A_3 = u_3 F'(u_o) + u_1 u_2 F''(u_o) + \frac{1}{3!}u_1^3 F'''(u_o),$$

$$A_4 = u_4 F'(u_o) + (u_1 u_3 + \frac{1}{2!}u_2^2)F''(u_o) + \frac{1}{2! u_1^2 u_2} F'''(u_o) + \frac{1}{4!}F^4(u_o),$$

Substituting equation (3.1.10) and (3.1.11) in (3.1.9) gives:

$$\sum_{n=0}^{\infty} y_n(x) = A - L^{-1}(g(x)) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right).. \qquad (3.1.12)$$

Now, to determine the component of $y_n(x)$, we use the Adomian decomposition method which makes use of the recursive relation:

$$y_o(x) = A + L^{-1}(g(x)),$$

$$y_{k+1}(x) = -L^{-1}(A_k), \quad k \geq 0, \tag{3.1.13}$$

Which gives:

$$y_o(x) = A + L^{-1}(g(x)),$$

$$y_1(x) = -L^{-1}(A_0),$$

$$y_2(x) = -L^{-1}(A_1),$$

$$y_3(x) = -L^{-1}(A_2),$$

$$y_4(x) = -L^{-1}(A_3),$$

$$\vdots$$

$$y_{k+1}(x) = -L^{-1}(A_k),$$

From the recursive relation above, the components of $y_n(x)$ can be determined and hence the series solution obtained.

For numerical purposes, the n-term approximation is given by,

$$\ddot{}_n = \sum_{j=0}^{n-1} y_j,$$

Where the series obtained will converge to that of the exact solution given, then

$$y(x) = \lim_{n\to\infty} \sum_{n=0}^{n-1} y_n(x), \tag{3.1.14}$$

In this chapter, this method will be analysed and we will prove its convergence

3.2. Convergence of the ADM

In this section, the order and rate of convergence will be defined and also, the convergence of the ADM will be proved.

i. Order and rate of convergence.

ii. Definition. For every $n \in NU\{0\}$, we define the rate of convergence as:

$$\sim_n = \begin{cases} \frac{\|y_{n+1}\|,}{\|y_n\|} \|y_n\| \neq 0, \\ 0, \|y_n\| = 0, \end{cases} \quad (3.2.1)$$

4.1.3 Theorem let N be an operator from a Hilbert space H into H and y be the exact solution of . Then the sum, $\sum_{n=0}^{\infty} y_n$, which is obtained from (3.1.12), converges to y then

$\exists 0 \leq \sim < 1, \|y_{n+1}\|, \forall_n \in Nu\{0\}$

3.2.4 Corollary. The expression in theorem (3.2.3) i.e. $\sum_{n=0}^{\infty} y_n$ converges to an exact solution y, when 0

$0 \leq \sim_n < 1, n = 1,2,3...$

Proof. A detailed proof of the convergence of ADM using the entire series property as shown by

Consider a non-linear functional equation

$$y = N(y) + f, \quad (3.2.2)$$

Since y(x) can be written as a series in an infinite form

$$y = \sum_{n=0}^{\infty} y_n.$$

This expression can be written by introducing ª, the ª is used to test for the convergence so that we have:

$$y_{ª} = \sum_{n=0}^{\infty} y_n ª^n \ . \quad (3.2.3)$$

Let's suppose that the convergence radius ¥ of (3.2.3) is greater than 1i.e¥ >1, we then conclude that (3.1.4) converges when $|ª| \leq ¥$.

Assume N(y) can be written in an entire series form:

$$N(y) = \sum_{i=0}^{\infty} g_i y^i \ , \quad (3.2.4)$$

With radius p>1, so that the series converges for $|y| < p.$ if we let $P = +\infty$, it is possible that the non-linear function N(y) converges for any y with $|y| < +\infty$.

Suppose we let:

$$¥ = p(y) = a_o + a_1 y + a_2 y^2 + \ldots + a_n y^n + \ldots, \qquad (3.2.5)$$

Be a series which converges for $|y| < +\infty$, and we also let

$$y = q(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots \quad , \qquad (3.2.6)$$

Be another series which converges for $|x| \le k, (k > 1)$

Substituting equation (3.2.6) into (3.2.5) we have:

$$¥ = a_o + a_1(b_o + b_1 x + b_2 x^2 + \ldots) + a_2(b_o + b_1 x + b_2 x^2 + \ldots)^2 + \ldots$$
$$= a_o + a_1 b_o + a_1 b_1 x + a_1 b_2 x^2 + a_2 b_o^2 + 2a_2 b_o b_1 (x) + a_2 b_1 x^2 + a_2 2b_o b_2 x^2 + \ldots,$$

So that we have:

$$\begin{cases} c_o = a_o + a_1 b_o + a_2 b_o^2 + \ldots, \\ c_1 = a_1 b_1 + 2a_2 b_1 b_o + \ldots, \\ c_2 = a_1 b_2 + a_2 (b_1^2 + 2b_o b_2) + \ldots, \end{cases} \qquad (3.2.7)$$

Which can be written in series $\sum_{i=0}^{\infty} c_n x^n$, the radius of convergence of this series is strictly greater than 1 i.e. the series converges for $|\pm| > 1$.

Putting these result into N(y) = $\sum_{i=0}^{\infty} g_i y^i$, since

$y = \sum_{n=0}^{\infty} y_n ª^n and$ N(y) = $\sum_{i=0}^{\infty} g_i y^i$, which can be written as N(y) = $\sum_{i=0}^{\infty} g_i (\sum_{n=0}^{\infty} y_n ª^n)^i$

So that we obtain a series

$$N(y) = \sum_{i=0}^{\infty} A_i ª^i \qquad (3.2.8)$$

Which converges for $|ª| \le p, (p > 1)$, where the A_i's depend on the u $u_o, u_1, \ldots, u_n$ which are determined from the relationship in (3.2.7).

Now substituting (3.2.3) and (3.2.4) in to (3.2.2), and assume that a = 1 so that we obtain:

$$\sum_{i=0}^{\infty} y_i - \sum_{i=0}^{\infty} A_i = f. \qquad (3.2.9)$$

This expression is satisfied by the Adomian's equation

$$\begin{cases} y_o = f, \\ y_1 = A_o, \\ \vdots \\ y_n = A_{n-1} \\ \vdots \end{cases} \qquad (3.2.10)$$

The proof is concluded by checking that the non-linear function has a series solution which remains stable under perturbation i.e. $\sum_{n=0}^{\infty} (1+\in)^n |y_n| < \infty, \ for \in > 0$ very small.And also that the non-linear operator N(y) can be developed in series according to y.

4. APPLICATION ON SOME ILLUSTRATIVE EXAMPLES

In this chapter, numerical experiments are performed on some illustrative examples of some linear and non-linear ordinary differential equations to check the difference between the methods described in the previous chapters. The performance of the methods are checked in terms of accuracy, efficiency and convergence.

Let's consider the following differential equations;

Example 1

Consider the following linear differential equation;

$$y' = 2xy, \quad y(o) = 1,$$

PIM solution. Using equation, the PIM iterates are obtained from,

$$y_o(x) = 1,$$

$$y_{n+1}(x) = y_o + \int_{x_o}^{x} f(t, y_n(t))dt.$$

Thus, the first four solutions are given by

$$y_o = 1,$$

$$y_1 = 1 + x^2,$$

$$y_2 = 1 + x^2 + \frac{1}{2}x^4,$$

$$y_3 = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6,$$

$$y_4 = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8,$$

ADM solution. Using equation (3.1.13), the ADM iterates are obtained from

$$y_o(x) = A,$$
$$y_{n+1}(x) = L^{-1}(A_n), n \geq 0,$$

Thus, the first four solutions are given by;

$$y_o = 1,$$
$$y_1 = x^2,$$

$$y_2 = \frac{1}{2}x^4,$$

$$y_3 = \frac{1}{6}x^6,$$

$$y_4 = \frac{1}{24}x^8.$$

Thus, the approximate solution for y(x) obtained using n = 0, 1, 2, · · · , 7 is gives;

$$¤_8(x) = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{120}x^{10} + \frac{1}{720}x^{12} + \frac{1}{5040}x^{14} .$$

Table 1. approximation of y(x) for the PIM and ADM at selected values of x and orders.

x	2nd order	4th order	6th order	8th order	exact
0.2	1.04080000	1.04081077	1.04081078	1.04081078	1.04081077
0.4	1.17280000	1.17350997	1.17351087	1.17351087	1.17351087
0.6	1.42480000	1.43327584	1.43332925	1.43332941	1.43332942
0.8	1.84480000	1.89548117	1.89647140	1.89648083	1.89648088
1.0	2.50000000	2.70833333	2.71805556	2.71827877	2.71828183
1.2	3.47680000	4.15362304	4.21760432	4.22061032	4.22069582
1.4	4.88080000	6.75063477	7.07042186	7.09787117	7.09932706
1.6	6.83680000	11.42257237	12.72976953	12.91849180	12.93581732
1.8	9.48880000	19.74915424	24.33125385	25.37612048	25.53372175
2.0	13.00000000	34.33333334	48.55555556	53.43174603	54.59815003
Run time	0.014	0.016	0.016	0.017	

Table 2. Error Estimation of y(x) using PIM and ADM for selected values of x and orders.

x	2nd order	4th order	6th order	8th order
0.2	0.00001077	0.00000000	0.00000001	0.00000001
0.4	0.00071087	0.00000090	0.00000000	0.00000000
0.6	0.00852942	0.00005358	0.00000017	0.00000001
0.8	0.05168058	0.00099970	0.00000948	0.00000025
1.0	0.21828183	0.00994849	0.00022627	0.00000306
1.2	0.74389582	0.06707278	0.00309150	0.00008550
1.4	2.21852706	0.34869229	0.02890520	0.00145589
1.6	6.09901732	1.51324495	0.20604779	0.01732552
1.8	16.04492175	5.78456751	1.2024679	0.15760127
2.0	41.59815003	20.26481669	6.04259447	1.16640400

Table 1 shows the comparison between results of the PIM, ADM and the exact solutions computed for different orders and selected values of x and their various run times. The two methods were combined because they generated the same results when computed. Table 2 gives the absolute errors between the PIM, ADM and the exact results. We can see from table 2, that the error rate are minimal for small values of x, accuracy of the ADM and PIM results generally increases when the number of iterations is increased, we can then say that they both converge faster to the exact solutions. In figure 1, we can see that the three methods gives close approximations, which confirms that the PIM and ADM are almost the same and they converge to the exact solutions. We therefore infer that the two methods are approximately the same for this problem.

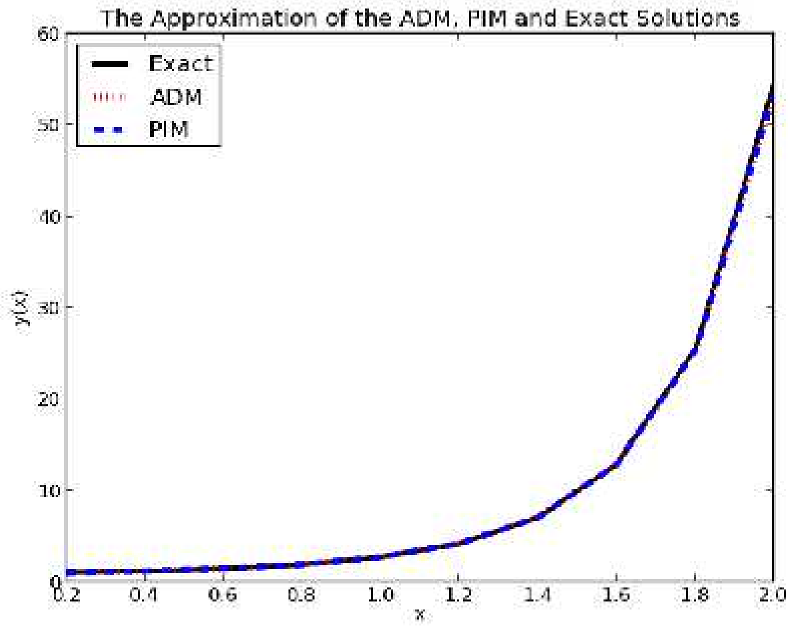

Figure 1. Approximation of y(x) for the ADM, PIM and the exact solution at the selected values of x considering the 8th order.

Example 2

We will now consider the following non-linear differential equation;

$$y' = y^2, \quad y(0) = 1.$$

With exact solution of $y(x) = \frac{1}{1-x}$.

PIM solution. Using the same approach as that of example 1 above. The first three solution of the iteration are given as;

$$y_o = 1,$$
$$y_1 = 1 + x,$$
$$y_2 = 1 + x + x^2 + \frac{1}{3}x^3,$$
$$y_4 = 1 + x + x^2 + x^3 + \frac{2}{3}x^4 + \frac{1}{3}x^5 + \frac{1}{9}x^6 + \frac{1}{63}x^7.$$

ADM solution. Using the same approach as that of example 1 above , the first three solutions are;

$$y_o = 1,$$
$$y_1 = x,$$
$$y_2 = x^2,$$
$$y_3 = x^3,$$

Hence, the approximate solution for y(x) obtained using n=0,1,2,….,7 is given by;

¤$_8(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7$

Table 3. Approximation of y(x) for the PIM selected values of x and orders.

X	2nd order	4th order	6th order	8th order	exact
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.11033333	1.11110938	1.11111111	1.11111111	1.11111111
0.2	1.24266667	1.24992670	1.24999965	1.25000000	1.25000000
0.3	1.39900000	1.42781460	1.42856219	1.42857136	1.42857143
0.4	1.58133333	1.66219250	1.66655381	1.66666502	1.66666667
0.5	1.79166667	1.98007633	1.99906325	1.99997461	2.00000000
0.6	2.03200000	2.42378080	2.49363107	2.49969413	2.50000000
0.7	2.30433333	3.06008988	3.29319138	3.32991902	3.33333333
0.8	2.61066667	3.99493989	4.73537940	4.95796008	5.00000000
0.9	2.95300000	5.39676995	7.72314683	9.22167456	10.00000000
Run time	0.031	0.131	1.147	2.262	

Table 4. Approximation of y(x) for the ADM selected values of x and orders.

X	2^{nd} order	4^{th} order	6^{th} order	8^{th} order	exact
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.11000000	1.11110000	1.11111100	1.11111111	1.11111111
0.2	1.24000000	1.24960000	1.24998400	1.25000000	1.25000000
0.3	1.39000000	1.42510000	1.42825900	1.42854331	1.42857143
0.4	1.56000000	1.64960000	1.66393600	1.66622976	1.66666667
0.5	1.75000000	1.93750000	1.98437500	1.99609375	2.00000000
0.6	1.96000000	2.30560000	2.43001600	2.47480576	2.50000000
0.7	2.19000000	2.77310000	3.05881900	3.19882131	3.33333333
0.8	2.44000000	3.36160000	3.95142400	4.32891136	5.00000000
0.9	2.71000000	4.09510000	5.21703100	6.12579511	10.00000000
Run time	0.017	0.032	0.32	0.044	

Tables 3 and.4 shows the comparison between results of the PIM, ADM and the exact solutions computed for different orders and some selected values of x and their run times. The absolute error between the PIM and the exact solutions; the ADM and the exact solutions are shown in tables 5 and 6 respectively. Figure 2 shows the comparison between the PIM, ADM and the exact solution at the 8th order. From table 5, we can see that the error rate are minimal for small values of x, accuracy and convergence of the PIM results generally increases when the number of iterations is increased. We can see from table 6 that the error approximation is large for the 2nd and 4th order, but on getting to the 8th order, the error is minimal for small values of x. We can then say that the ADM is less accurate and converges slower to the exact solutions. When comparing between the two methods using figure 2, we can see that the PIM is more accurate and converges to the exact solutions faster than the ADM, but the ADM is more efficient than the PIM since the run time estimate for the PIM is much more greater than that of the ADM.

Table 5. Error Estimation of y(x) using PIM selected values of x and orders.

x	2^{nd} order	4^{th} order	6^{th} order	8^{th} order
0.0	0.00000000	0.00000000	0.00000000	0.00000000
0.1	0.00077778	0.00000173	0.00000000	0.00000000
0.2	0.00733333	0.00007330	0.00000035	0.00000000
0.3	0.02957143	0.00075682	0.00000924	0.00000007
0.4	0.08533333	0.00447416	0.00011286	0.00000165
0.5	0.20833333	0.01992367	0.00093675	0.00002539
0.6	0.46800000	0.07621920	0.00636893	0.00030587
0.7	1.02900000	0.27324345	0.04014196	0.00341431
0.8	2.38933333	1.00506011	0.26462060	0.04203992
0.9	7.04700000	4.60323005	2.27685317	0.77832544

Table 6. Error Estimation of y(x) using ADM the selected values of x and orders.

x	2^{nd} order	4^{th} order	6^{th} order	8^{th} order
0.0	0.00000000	0.00000000	0.00000000	0.00000000
0.1	0.00111111	0.00001111	0.00000011	0.00000000
0.2	0.01000000	0.0004000	0.00001600	0.00000064
0.3	0.03857143	0.00347143	0.00031243	0.00002812
0.4	0.10666667	0.01706667	0.00273067	0.00043691
0.5	0.25000000	0.06250000	0.01562500	0.00390625
0.6	0.54000000	0.19440000	0.06998400	0.02519424
0.7	1.14333333	0.56023333	0.27451433	0.13451202
0.8	2.56000000	1.63840000	1.04857600	0.67108864
0.9	7.29000000	5.90490000	4.78296900	3.87420489

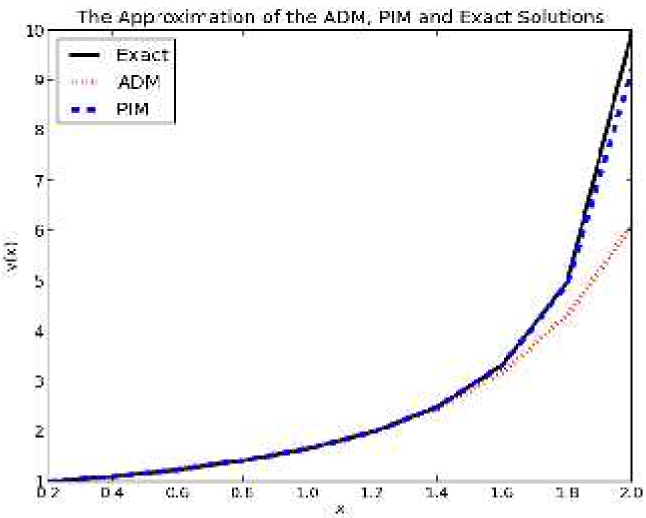

Figure 2. Approximation of y(x) for the ADM, PIM and the Exact Solution at the selected values of x considering the 8th order.

Example 3

$$y' + y^2 = 1. \qquad y(0) = 0.$$

With exact solution of $y(x) = \dfrac{1-e^{-2x}}{1+e^{-2x}}$.

PIM solution. The first four solutions are given by;

$$yo = 0,$$
$$y_1 = x,$$
$$y_2 = x - \frac{1}{3}x^3,$$
$$y_3 = x - \frac{1}{3}x^3 - \frac{1}{63}x^7 + \frac{2}{15}x^{15},$$
$$y_4 = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{38}{2835}x^9 - \frac{134}{51975}x^{11} - \frac{1}{59535}x^{15}.$$

ADM solution. The first four solutions are

$$yo = x,$$
$$y_1 = -\frac{1}{3}x^3,$$
$$y_2 = \frac{2}{15}x^5,$$
$$y_3 = -\frac{17}{315}x^7,$$
$$y_4 = \frac{62}{2835}x^9.$$

Hence, The approximate solution for y(x) obtained using n = 1,2,3,…,7 is given by;

$$\text{¤}_8(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{62}{2835}x^9 - \frac{1382}{155925}x^{11} + \frac{21844}{6081075}x^{13} - \frac{929569}{638512875}x^{15}.$$

Tables 7 and 8 shows the comparison between results of the PIM, ADM and the exact solutions computed for different orders and some selected values of x and their run times. The absolute error between the PIM and the exact solutions; the ADM and the exact solutions are shown in tables 9 and 10 respectively. Figure 3 shows the comparison between the PIM, ADM and the exact solution at the 8th order. From table 9, we can see that the error rate are minimal for small values of x, accuracy and convergence of the PIM results generally increases when the number of iterations is increased. We can see from table 10 that the error approximation is large for the 2nd and 4th order, but on getting to the 8th order, the error is minimal for small values of x. From figure 3, we can then say that the ADM is less accurate and converges slower to the exact solutions, in fact, it Converges when $0 < x < 1$ but when $x > 1$ it diverges. When comparing between the two methods,

We can infer that the PIM is more accurate and converges to the exact solutions faster than the ADM for this problem but the ADM is more efficient than the PIM since the run time estimate for the PIM is much greater than that of the ADM.

Table 7. Approximation of y(x) for the PIM selected values of x and orders.

X	2nd order	4th order	6th order	8th order	Exact
0.2	0.19733333	0.19737532	0.19737532	0.19737532	0.19737532
0.4	0.37866667	0.37994699	0.37994896	0.37994896	0.37994896
0.6	0.52800000	0.53698338	0.53704935	0.53704957	0.53704957
0.8	0.62933333	0.66330092	0.66402980	0.66403673	0.66403677
1.0	0.66666667	0.75716627	0.76150059	0.76159305	0.76159416
1.2	0.62400000	0.81562802	0.83293984	0.83363872	0.83365461
1.4	0.48533333	0.82929787	0.88164769	0.88521413	0.88535165
1.6	0.23466667	0.77889309	0.90719341	0.92084260	0.92166855
1.8	-0.14400000	0.63789308	0.90121266	0.94303954	0.94680601
2.0	-0.66666667	0.39169207	0.84363640	0.95067963	0.96402758
	0.034	0.096	1.052	2.593	

Table 8. Approximation of y(x) for the ADM selected values of x and orders

X	2nd order	4th order	6th order	8th order	Exact
0.2	0.19737600	0.19737532	0.19737532	0.19737532	0.19737532
0.4	0.38003200	0.37994931	0.37994896	0.37994896	0.37994896
0.6	0.53836800	0.53707763	0.53705016	0.53704958	0.53704957
0.8	0.67302400	0.66464131	0.66407744	0.66403951	0.66403677
1.0	0.80000000	0.76790123	0.76263013	0.76176432	0.76159416
1.2	0.95577600	0.87523975	0.84781849	0.83847884	0.83365461
1.4	1.20243200	1.08538025	1.01157026	0.96499619	0.88535165
1.6	1.63276800	1.68692852	1.74544229	1.80843315	0.92166855
1.8	2.37542400	3.40939561	5.19301064	8.26849728	0.94680601
2.0	3.60000000	7.88924162	19.164044813	48.79534367	0.96402758
Run time	0.014	0.016	0.023	0.041	

Table 9. Error Estimation of y(x) using PIM selected values of x and orders.

x	2nd order	4th order	6th order	8th order
0.2	0.00004199	0.00000000	0.00000000	0.00000000
0.4	0.00128229	0.00000197	0.00000000	0.00000000
0.6	0.00904957	0.00006619	0.00000022	0.00000000
0.8	0.03470344	0.00073585	0.00000697	0.00000004
1.0	0.09492749	0.00442789	0.00009357	0.00000111
1.2	0.20965461	0.01802659	0.00071477	0.00000159
1.4	0.40001832	0.05605378	0.00370396	0.00013752
1.6	0.68700188	0.14277546	0.01447514	0.00082595
1.8	1.09080601	0.30891293	0.04559335	0.00376647
2.0	1.63069425	0.572335519	0.12039118	0.01334795

Table 10. Error Estimation of y(x) using ADM selected values of x and orders.

x	2nd order	4th order	6th order	8th order
0.2	0.00000068	0.00000000	0.00000000	0.00000000
0.4	0.00008304	0.00000035	0.00000000	0.00000000
0.6	0.00131843	0.00002806	0.00000060	0.00000001
0.8	0.00899723	0.00060454	0.00004067	0.00000274
1.0	0.03840584	0.00630708	0.00103597	0.00017016
1.2	0.12212139	0.04158514	0.01416389	0.00482423
1.4	0.31708035	0.20002860	0.12621861	0.07964454
1.6	0.71109945	0.76525997	0.82377373	0.88676459
1.8	1.42861799	2.46258960	4.24620463	7.32169126
2.0	2.63597242	6.92521404	18.20002055	47.83131609

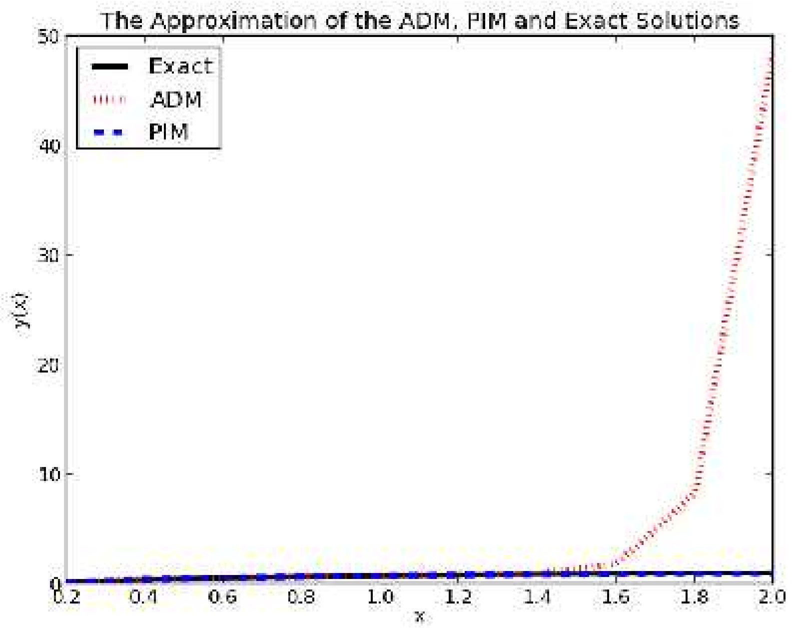

Figure 3 Approximation of y(x) for the ADM, PIM and the Exact Solution at the selected values of x considering the 8th order

5. SUMMARY

In this work, we presented two analytical series based methods which solve both non-linear and linear initial value problems. The methods considered are the Picard iteration method (PIM) and the Adomian decomposition method (ADM).

The application of these methods were illustrated by solving some linear and non-linear differential equations, and their performance was checked in terms of efficiency, accuracy and convergence. For the linear equations, when considering the accuracy and convergence rates, it was found that the two methods are approximately the same. On the other hand, for the non-linear differential equations, we found that the PIM was more accurate and converges faster than the ADM, in fact, the ADM converges at the region $0 < x < 1$, but when $x > 1$, it diverges. In terms of computational efficiency, the PIM and ADM was found to have the same efficiency rate for the linear differential equations but the ADM was found to be better than the PIM in the non-linear differential equations because the implementation of the ADM algorithm took less time than the PIM algorithm. From the study we figured out that the limitations of the ADM are that it converges for $x < 1$, the PIM is not computationally efficient as it takes a long time to run on Maple and for a large number of terms e.g. $n > 8$, the computations become too much for the computer to handle and it crashes. Therefore, we conclude that both ADM and PIM are good techniques for solving linear differential equations but for non-linear differential equations, the ADM is computationally efficient but less accurate than the PIM.

For future work, attempts will be made to solve the full governing equations without making any assumptions. The equations will be solved numerically with the aid of new spectral method based hybrids of the ADM/PIM that will be developed in the study. The research will start with a review of the existing literature on the ADM/PIM and the identification of the limitations and weaknesses of these studies. Numerical simulations will be conducted to obtain the solution of selected problems that will be considered as numerical experiments. The results from the numerical simulations will be validated against results that exist in literature. Numerical simulation and computing work will be done.

Some of the advantages of this method are that it converges fast to the exact solution. it requires less computational work than the other methods and it also has the ability to solve non-linear problems without line arising.

Some of the disadvantages of this method are that it gives a long series solution which must be cut short for it to be useful in practical application and the convergence rate for wider region is said to be slow.

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