SOME RESULTS ON GENERALIZED REVERSE DERIVATIONS ON PRIME NEAR RINGS WITH SEMIGROUP IDEALS

M.Sc. THESIS

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November 2022

HARAMAYA UNIVERSITY, HARAMAYA

Some Results on Generalized Reverse Derivations on Prime Near Rings with Semigroup Ideals

A Thesis Submitted to Department of Mathematics,

Postgraduate Program Directorate

HARAMAYA UNIVERSITY

In Partial fulfillment of the Requirements for the degree of

MASTER OF SCIENCE IN MATHEMATICS

(ALGEBRA)

Tadesse Yacoob

November 2022

Haramaya University, Haramaya

Postgraduate Program Directorate

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DEDICATION

I dedicate this thesis manuscript to my beloved mother Bekelech Megiso and my father Yacoob Leramo who provided everything needed to progress in my life.

STATEMENT OF THE AUTHOR

By my signature below, I declare that this thesis is my own work. I have followed all ethical and technical principles of scholarship in the preparation, and compilation of this thesis. Any scholarly matter that is included in the thesis has been given recognition through citation.

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Notations

[x, y]	Multiplicative Commutator
Z(N)	Multiplicative Centre
(x, y)	Additive Commutator
(f, d)	Generalized Reverse Derivation

BIOGRAPHICAL SKETCH

The author was born on November 1988 in SNNPR state, Hadiya Zone, West Badewacho woreda, Kotto Keble from his mother Bekelech Megiso and father Yacoob Leramo. He attended his primary education at Kotto Primary School. Then after, he joined Shone secondary and Preparatory School to attend his secondary education. Then joined Hawassa University in 2011 and received Bachelor of Science degree in Mathematics in June, 2013. After graduation he has been employed in SNNPR state, Hadiya Zone, west Badewacho woreda, Danema secondary and preparatory School. In 2019, he joined postgraduate program at Haramaya University, College of Natural and Computational Science, Department of Mathematics to pursue a program of study for M.Sc. degree in Mathematics (Algebra).

ACKNOWLEDGEMENT

First of all, I would like to thank my Almighty God for giving his grace and blessing, starting up to final write up of this thesis. My profound appreciation and thanks go to my thesis major advisor **Dr. Phool Miyan** and my thesis co-advisor **Dr. Seleshi Demie** who had been kindly and extensively advised me by giving constructive comments and to make improvement on this thesis. I wish to extend my sincere thanks to all my family and my beloved mother Bekelech, who have been constant source of encouragement and support. Finally, I would like to express my heartfelt thanks to West Badewacho Woreda Education office and Haramaya University Mathematics department who give me encouragement and help me during this study.

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ABSTRACT

The principal purpose of this thesis is to extend results from reverse derivation prime rings to generalized reverse derivations on prime near rings in the setting of semigroup ideals and discuss conditions under which a prime near ring is commutative ring. The extensions and conditions are not easily generalized, because one of a classical problem in near ring theory is to study and generalize the conditions under which conditions a prime near ring is commutative ring. First important preliminary concepts, examples, lemmas and theorems are presented. In order to study generalized reverse derivations on prime near rings with semigroup ideals with some terms such as derivations, reverse derivations, generalized derivations are discussed. The concept of generalized reverse derivations in prime near rings with semigroup ideals and some important results of extension of generalized reverse derivations in prime near rings with semigroup ideals were presented and we have showed that certain conditions involving generalized reverse derivations forces a prime near ring to be a commutative ring. Moreover, we have proved that a non-zero generalized reverse derivation (f, d) on N with semigroup ideals U of N such that f acts as a homomorphism and an anti-homomorphism on semigroup ideal U of N, thus f is the identity map on semigroup ideals U of N. Finally, we have proved that product of two generalized reverse derivations act as generalized reverse derivation on semigroup ideal ideals of N, such that at least one of generalized reverse derivation is zero. For further study researchers can do on multiplicative (generalized) reverse derivations on prime near ring.

Keywords: *Prime Near Ring, Semigroup Ideal, commutativity, Generalized Derivation, Reverse Derivation, Generalized Reverse Derivation.*

1. INTRODUCTION

1.1. Background of the Study

Dickson (1905) has done an axiomatic research to primary step towards near rings. The theory of near rings is now a sophisticated theory which has found numerous applications in various areas. Dickson showed that there do exist "fields with only one distributive law" (Named near fields). Near fields showed up to be useful in coordinating certain important classes of geometric planes. Connections between other parts of near rings (especially near fields) and geometry come up at several places. Efficient block designs and codes can be constructed from finite near-rings. Many parts of the well-established theory of rings were transferred to near rings and new near ring precise features were discovered, constructing up a theory of near rings step by step.

Near rings are generalizations of rings. It is natural to generalize various concepts of rings to near rings. Beidleman (1967), Ligh and Luh (1976), Clay (1968), Bell (1970) and others have generalized various concepts of near rings. The concept of near rings first introduced by Pilz (1983). The study of derivation was initiated during the 1950s and 1960s. Despite the concept of derivation in rings being quite old and playing a significant role in various branches of mathematics, however, ring with derivation has been studied by many authors specially the relationships between derivations and the structure of rings.

Nowicki (1987) studied the fundamental relations between the operation of differentiation (derivation) and that of addition and multiplication of functions have been known for a long time as the notion of the derivative itself. The relations were depending that the operation of differentiation of functions on the smooth varieties with respect to a given tangent field not only has the formal properties of differentiation but also conversely; the tangent field fully characterized by such an operation.

The study of derivations in rings studied long back, but got impetus only after Posner (1957) established two very striking results on derivations in prime rings. The results under reference state that; (i), In a 2-torsion-free prime ring, if the iterate of two derivations is a derivation, then one of them must be zero; (ii), A prime ring R admitting a nonzero centralizing derivation d must be commutative. A number of authors have extended these 2 results by considering mapping which is only assumed to be centralizing on an appropriate subset of the ring. Awtar (1973) considered centralizing derivations on Lie and Jordan ideals. In the Jordan case, Awtar proved that if a prime ring of characteristic not two has a nontrivial derivation which is centralizing on a Jordan ideal, then the ideal must be central.

Several authors have studied about derivations on rings. Bell and Kappe (1989), and Rahman (2002), and Bell and Daif (1985) studied about the derivations and Commutativity of prime rings. Motivated by the concept of derivation in rings Bell and Mason (1987) initiated the study of derivations in near rings. A mapping $d: N \to N$ is said to be a derivation on a near ring N if (i), $d(x + y) = d(x) + d(y)$ and (ii), $d(xy) = xd(y) + d(x)y$ holds for all $x, y \in N$. It was shown by Wang (1994) that condition (ii). is equivalent to $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$, which facilitates the study of derivations in near rings.

Bresar (1991) introduced the concept of generalized derivation in rings as follows: Let S be a non-empty subset of R. An additive map $f: R \to R$ is said to be generalized derivation on S if there exist derivation $d: R \to R$ such that $f(xy) = f(x)y + xd(y)$ holds for all $x, y \in S$. This concept covers the concept of derivation already known to us for ring theory. Later the study was done by Hvala (1998) and Golbasi (2006) and et al. about generalized derivations in the setting of prime rings and semiprime rings and several known results for derivation in prime and semiprime rings were extended in the setting of generalized derivations in rings by the above authors.

Motivated by the above concept, Golbasi (2006) introduced the concept of generalized derivations in near rings. Later Bell (2008) also studied this notion and derived some commutativity theorems of prime near rings equipped with generalized derivation. Herstein (1957) initiated the study of reverse derivations in prime rings, where he proved that a prime ring possessing a reverse derivation is a commutative integral domain and hence reverse derivation behaves like an ordinary derivation. Further Bresar and Vukman (1989) have studied the notion of reverse derivation and some properties of reverse derivations. Additionally, they proved that if a prime ring R admits a derivation respectively reverse derivation, then either $d = 0$ or R is commutative.

Aboubakr and Gonzalez (2015) studied the relationship between generalized reverse derivation and generalized derivation on an ideal in semiprime rings. An additive mapping $f: R \to R$ is a right generalized reverse derivation if there exists a derivation d from R to R such that $f(xy) = f(y)x + yd(x)$, for all x, y in R and f is a left generalized reverse derivation if there exists a derivation d from R to R such that $f(xy) = d(y)x + yf(x)$ for all x, y in R. Finally, f is a generalized reverse derivation of R associated with d if it is both right and left generalized reverse derivation of ring R. Very recently Tiwari et al. (2018) defined multiplicative (generalized) reverse derivation. A map $f: R \to R$ is said to be a multiplicative (generalized) reverse derivation if $f(xy) = f(y)x + yd(x)$ holds for all $x, y \in R$, where d is any map on R.

Reddy *et al*. (2015) defined that an additive mapping $f: R \to R$ is a right generalized reverse derivation if there exists a derivation $d: R \to R$ such that $f(xy) = f(y)x + yd(x)$ for all x, y in R and f is a left generalized reverse derivation if there exists a derivation $d: R \to R$ such that $f(xy) = d(y)x + yf(x)$ for all x, y in R. So that f is generalized reverse derivation of R associated with d if it is both right and left generalized reverse derivation of ring R.

1.2. Statement of the problem

The concept of reverse derivations has relations with some generalizations of derivations. Herstein (1957) studied concept of reverse derivation prime rings. An additive mapping d: R → R which satisfies d (xy) = d(y) x + yd(x) for all x, y ∈ R is called a reverse derivation on rings. Golbasi (2006) showed that N is a 2−torsion free prime near ring with center Z(N), (f, d) and (g, h) are two generalized derivations on N. In this case: If f ([x, y]) = 0 or f ([x, y]) = ± [x, y] or f ^{2}(x) ∈ Z(N) for all x, y ∈ N, then N is a commutative ring. Reddy *et al*. (2015) introduced the notion of reverse derivation on prime near rings and they extended ideas from reverse derivations on prime rings towards the generalized reverse derivations on prime near rings. They considered a prime near ring 2- torsion free with center Z(N) and two generalized reverse derivations (f, d) and (g, h) of N. They also Proved that if f ([x, y]) = 0 or f (yx) ± xy ∈ Z(N) or f (xy) ± xy ∈ Z(N) for all x, y ∈ N then N is commutative ring. Moreover they proved that a non-zero generalized reverse derivation (f, d) on N such that if f acts as a homomorphism and an anti-homomorphism, then f is the identity map.

We have extended that some results on reverse derivation on prime rings to generalized reverse derivations of prime near rings in the setting of semigroup ideals. We have took that N is 2-torsion free prime near ring with center Z(N) and U be a semigroup ideal of N. Our aim is to establish conditions under which a near-ring becomes a commutative ring. If (f, d) is a generalized reverse derivation on N associated with reverse derivation d on N in the setting of semigroup ideals of N satisfying that f ([x, y]) = 0, f (yx) ± xy ∈ Z(N) or f (xy) ± xy ∈ Z(N) for all x, y ∈ U, then N is commutative ring. Moreover, we have showed that (f, d) and (g, h) be two generalized reverse derivations of N, if (fg, dh) acts as a generalized reverse derivation on N in setting of semigroup ideals of N, then (N, +) is abelian and f = 0 $or g$= 0. Finally, we have proved that, if f acts as homomorphism and as anti– homomorphism on semigroup ideals U of N, then f is identity map on U.

1.3. Objectives

The main objective is to extend some results on reverse derivations on prime rings to generalized reverse derivations on prime near rings with semigroup ideals.

The study intended to explore the following specific objectives:

1. To describe the conditions of non-zero generalized reverse derivations on N with semigroup ideals and show that f is an identity map on U.
2. To discuss about essentiality of primness of N on generalized reverse derivations on prime near rings.
3. To determine the conditions on generalized reverse derivations on prime near rings with semigroup ideals U of N is commutative ring.
4. To provide proofs on some results on generalized reverse derivations on prime near rings in the setting of semigroup ideals.
5. Extend ideas from reverse derivations on prime rings to generalized reverse derivations on prime near rings in the setting of semigroup ideals of N.

2. LITERATURE REVIEW

2.1. Derivations on Prime Rings

Derivation has been studied by many authors for the last 60 years, especially the relationships between derivations and the structure of rings. One of the questions which is often appeared in algebra and analysis is whether a map can be defined by its local properties. Herstein (1976) posed the question whether a map, which acts like a derivation of prime rings, is induced by an ordinary derivation was a well-known. Derivations in prime rings and semiprime rings have been studied by several algebraists in various directions. Derivation in ring theory was introduced by Posner (1957). In the process of improving the derivations in ring theory, there are various derivations such as generalized derivation, Jordan derivation, symmetric bi-derivation, and generalized Jordan derivation. (Tiwari *et al., 2018)* defined multiplicative (generalized) reverse derivation. A map f: R $\rightarrow$ R is said to be a multiplicative (generalized) reverse derivation if $f(xy) = f(y)x + yd(x)$ holds for all x, y $\in$ R, where d is any map on R.

2.2. Derivations on Prime Near Rings

An attempt to present an up-to-date account of work on derivations and its various invariants in the setting of near-rings. The work has been presented in a manner suitable for everybody who has some basic knowledge in near-ring theory. Bell and Mason (1987) introduced the concept of derivation in near-rings as follows that a derivation ′d ′ on N is defined to be an additive mapping d: N →N satisfying the product rule $d(xy) = xd(y) + d(x)y$ for all x, y ∈ N and proved that if a 2-torsion free zero-symmetric prime near ring N admits a non-zero derivation D for which D(N) ⊆ Z(N), then N is a commutative ring.

Golbasi (2006) showed that if N be a prime near ring and N admits a non-zero derivation d such that d([x, y]) = 0, (i), d[x,y] = [d(x),y] (ii), [d(x),y] = [x,y] this above (i) and (ii) are equivalent for all x, y ∈ N, then N is a commutative ring. Wang (1994) obtained derivation on near a ring which is as follows: Let N be a 2-torsion free prime ring and d_1, d_2 be derivations of N such that d_1d_2 is also a derivation. Then $[d_1\{x), d_2\{x)] = 0$ for all x, y ∈ N if and only if either $d_1= 0$ or $d_2=0$.

Ali *et al*. (2013) defined an additive mapping d: N → N is a derivation on N if d (xy) = d(x) y + xd(y), for all x, y ∈ N. An additive mapping f: N→N is said to be a right (resp. left) generalized derivation with associated derivation d if f (xy) = f (x)y +xd(y) (resp. f (xy) = d(x)y + x f(y)), for all x, y ∈ N, and f is said to be a generalized derivation with associated derivation d on N if it is both a right generalized derivation and a left generalized derivation on N with associated derivation d and proved that let N be a prime near ring and U is a non-zero semigroup ideal of N. If f is a non-zero left generalized derivation with associated non-zero derivation d such that f ([x, y]) = 0, f ([x, y]) = [x, y] or f ([x, y]) = −[x, y] for all x, y ∈ U, then N is a commutative ring.

2.3. Reverse Derivations on Prime Near Rings

Herstein (1957) studied the notion of reverse derivation has relations with some generalizations of derivations. Each reverse derivation is a Jordan derivation (but the converse is not true in general). Bresar and Vukman (1989) have introduced the notion of a reverse derivation as an additive mapping d from a ring R into itself satisfying d(xy) = d(y)x+ yd(x), for all x, y ∈ R. Obviously, if R is commutative, then both derivation and reverse derivations are the same. Aboubakr *et al.* (2015) generalized the concept of reverse derivation to generalized reverse derivation and provided a study of relationship between generalized reverse derivations and generalized derivations.

Golbasi (2006) studied the notion of generalized derivation in near-rings as follows: An additive mapping f: N → N is called a right generalized derivation with associated derivation d if f(xy) = f(x)y + xd(y), for all x, y ∈ N and f is called a left generalized derivation with associated derivation d if f(xy) = d(x)y + xf(y), for all x, y ∈ N so f is called a generalized derivation with associated derivation d if it is both a left as well as a right generalized derivation with associated derivation d.

Reddy *et al.* (2015) defined that an additive mapping f: R →R is a right generalized reverse derivation if there exists a derivation d: R→R such that f(xy) = f(y)x + yd(x) for all x,y in R and f is a left generalized reverse derivation if there exists a derivation d: R→R such that f(xy) = d(y)x + yf(x), for all x, y in R. Finally, f is a generalized reverse derivation of R associated with d if it is both right and left generalized reverse derivation of R. Let (f, d) be a generalized reverse derivation on N where d is nonzero derivation. If f ([x, y]) = 0, f(yx) ± xy ∈ Z(N) (or) f(xy) ± xy ∈ Z(N), for all x, y ∈ N, then N is commutative ring.

3. MATERIALS AND METHODS

Sources in the website and Haramaya University library were used to collect all the pieces of information about generalized reverse derivations on prime near rings with semigroup ideals and recorded subsequently.

Specifically,

- Relevant journals and books were consulted to gather information about generalized reverse derivation on prime near rings with semigroup ideals.
- Collected information's were arranged.
- Important preliminary concepts, definitions, and theorems were discussed to make the concepts clear.
- Some theorems and lemmas on generalized reverse derivation on prime near rings with semigroup ideals with detail explanatory steps to make the concept clear.
- Important lemmas were used to prove theorems.

4. PRELIMINARY

In this chapter, we deal with basic definitions, examples and lemmas which have been important ideas and concepts for the main results of chapter 5. We begin with the following definitions.

Near Ring

Definition 4.1 (Pilz, 1983) A right near ring is a non-empty set N together with two binary operations usually called addition '+' and multiplication ' ·' such that

i) (N, +) is a group (not necessarily abelian)

ii) (N, ·) is a semigroup

iii) $(x + y) \cdot z = x \cdot z + y \cdot z$ holds for all x, y, z $\in$ N (right distributive law)

If instead of (iii), we have the left distributive law,

iii') $x \cdot (y + z) = x \cdot y + x \cdot z$ for all x, y, z $\in$ N holds, then

N is said to be a left near ring.

We consider right near ring throughout this thesis.

Example 4.1 (pilz, 1983) Let N = Z_3 = {0, 1, 2} with + and · given by the scheme: (0, 1, 2)

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Hence (N, +, ·) is a near ring.

Definition 4.2 (Ibraheem, 2018) A near ring N is called zero-symmetric right (left) near ring if 0.n (n.0) $=$ 0 for all n $\in$ N.

Example 4.2. 1. (Bell and Mason, 1987) A near-ring N is called zero symmetric if for all $x \in N$ satisfies $x \cdot 0 = 0$. The set of integers $\mathbb{Z}$ with the usual addition and multiplication operation then $(\mathbb{Z}, +,\cdot)$ is the zero symmetric near-ring. while the set of integers Z with the usual addition operations and multiplication operation defined as for all $x, y \in$ Z applies $x \cdot y = x$ then (Z ,+,·) is near-ring but not zero symmetric.

2. (Bell and Mason, 1997) the set of all polynomials over $\mathbb{Z}_2$ denoted by $\mathbb{Z}_2[x]$ with usual polynomial addition and multiplication operations, $(\mathbb{Z}_2[x], +, \cdot)$ is a prime and zero symmetry near ring.

4.1. Basic Concepts on Prime Near Rings

Definition 4.1.1 (Bell and Mason, 1987) A near ring N is said to be prime if xNy = {0} for all x, y $\in$ N implies x = 0 or y= 0.

Example 4.1.1 (Bell and Mason, 1987) Let (N,+) be a group (not necessarily abelian).

Define multiplication $*$ on N as follows:

$x*y = x$, for all x, y $\in$ N is a prime right near ring because whenever we take xNy = {0}

implies that x = 0 or y = 0.

Definition 4.1.2 (kon, 1988) An ideal P of a near ring N is called prime if AB $\subseteq$ P implies A $\subseteq$ P or B $\subseteq$ P, for any ideals A and B of near ring N is called a prime near ring if {0} is a prime ideal.

Example 4.1.2 If N is a constant near ring, then each normal subgroup of (N,+) is a prime ideal.

Verification: Suppose N is a constant near ring. Then nk = n for all n, k $\in$ N. Let M be a normal subgroup of (N, +). Let n $\in$ N. Since N is constant, we have mn = m $\in$ M for all m $\in$ M. Therefore, MN $\sqsubseteq$ M, and so M is a right ideal of N. Let n, n' $\in$ N and m $\in$ M. Now n(n' + m) – nn' = n – n = 0 $\in$ M (since N is constant). This shows that M is a left ideal of N. Therefore, M is an ideal of N. Next we show that M is a prime ideal of N. Let I and J be ideals of N such that IJ ⊑ M. Now I = IJ ⊑ M.. Hence, M is a prime ideal of N.

Example 4.1.3 Each constant near ring is a prime near ring.

Verification: we know that (0) is a normal subgroup of N. Since N is a constant near ring, by above example 4.1.2 it follows that (0) is a prime ideal. Thus, N is a prime near ring.

Example 4.1.4 (Prasad and Satyanarayana, 2013) every integral near ring is a prime near ring.

Verification: Suppose N is an integral near ring. To show that N is a prime near ring,

It is enough to show (0) is a prime ideal.

Let I and J be ideals of N such that IJ ⊆ (0).

if either I = (0) or J = (0), then there is nothing to prove.

If possible, suppose that I ≠ (0) or J ≠ (0),

Then we can choose 0 ≠ a ∈ I and 0 ≠ b ∈ J such that ab = 0.

This is a contradiction to the fact that N is integral,

therefore either I = (0) or J = (0).

Thus we have proved that (0) is a prime ideal of N.

Hence, N is a prime near ring.

Note: Every integral near ring is a prime near ring and each constant near ring is a prime near ring. Of course, N is a prime ideal of N, so {0} is a prime near ring.

Definition 4.1.3 (Saed, 2018) Let N be defined a near ring. The symbol Z(N) will denote the multiplicative center of N, that is Z(N) = {x ∈ N | xy = yx for all y ∈ N}

Definition 4. 1. 4 (Bharathi and Jayalakshmi, 2017) for all x, y ∈ N stated that

[x, y] $= xy - yx$ is said to be commutator and the symbol $(x \circ y)$ = xy + yx will stand for the anti-commutator.

Identities: (Bhanathi and Jayalakshmi, 2017) for any x, y, z $\in$ N, the following identities hold.

i. $[x, yz] = y[x, z] + [x, y]z$
ii. $[xy, z] = x[y, z] + [x, z]y$
iii. $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$
iv. $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$

4.2. Prime Near Rings with Semigroup Ideals

In this section we discussed, some basic definition and results for a prime near rings including semigroup ideal.

Definition 4.2.1 (Bell, 1997) A non-empty subset U of N is a semigroup right ideal (resp. semigroup left ideal) if $UN \subseteq U$ (resp. $NU \subseteq U$) and if U is both semigroup right ideal and semigroup left ideal, it is called a semigroup ideal.

Example 4.2.1 (Bell, 1997) Let N = {0, a, b, c} with addition and multiplication tables defined as below:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
c	0	a	c	c

If we take A = {0, a}, B = {0, a, b} and C = {0, a, c}, then B, C are semigroup right ideals of N and A is a semigroup ideal of N.

Definition 4.2.2 (Satyaranyana and Prasad, 2013)

Let N be a near ring

i. A non-zero element n is said to be a right zero divisor (left zero divisor, respectively) if there exists a non-zero element a $\in$ N such that an = 0(na = 0, respectively).
ii. A near ring N is said to be integral near ring if it has no zero divisors.

Lemma 4.2.1 (Bell, 1997) let N be a prime near ring and d be a non – zero derivation on N.

i. If U is a non-zero semigroup right ideal or a non-zero semigroup left ideal of N, then $d(U) \neq \{0\}$.
ii. If U is a non-zero semigroup right ideal of N and $x \in N$ which centralizes U, then $x \in Z(\mathrm{N})$

Lemma 4.2.2 (Bell and Mason, 1987) Let N be a prime near ring and

U be a semigroup ideal of N.

I. If z is a non-zero element in Z(N), then z is not a zero divisor.
II. If there exists a non-zero element z of Z(N) such that z+z $\in$ Z(N), then (N, +) is abelian.

Lemma 4.2.3 (Bell, 1997) Let N be a prime near-ring. if U is a non-zero semigroup right ideal (resp. semigroup left ideal) subset of N and x is an element of N such that Ux = {0} (resp. xU ={0}) then x = 0.

Proof: Since U ≠ {0}, there exist an element u ∈ U such that u ≠ 0.

Consider that uNx ⊆ Ux = {0}.

Since u ≠ 0 and N is a prime near-ring,

we have x = 0.

Lemma 4.2.4 (Bell, 1997) Let N be a prime near ring, and U is a non-zero semigroup ideal subset of N. If x, y ∈ N and xUy ={0}, then x = 0 or y = 0

Proof: Let x, y ∈ N and xUy = {0}. then xUNy ⊆ xUy = {0}.

Since N is a prime near-ring, xU =0 or y = 0.

If y = 0, then we are done.

so if y ≠ 0, then xU = 0.

applying Lemma 4.2.3, x = 0.

Lemma 4.2.5 (Argac, 2004) Let N be a prime near-ring and U be a non-zero semigroup ideal of N. If (U, +) is abelian, then (N,+) is abelian.

Proof: Since (U,+) is abelian, we have $u + v = v + u$, for all $u, v \in U$.

Taking xu instead of u and yu instead of v, where $x, y \in N$.

We obtain $xu + yu = yu + xu$.

Then, we get $(x + y - x - y)\, u = \{0\}$, for all $u \in U$, and $x, y \in N$.

It means that $(x, y)\, U = \{0\}$. Since U is a semigroup ideal of N,

We get $(x, y)\, NU = \{0\}$.

Since N is prime near-ring then either $(x, y) = 0$ or $U = 0$ but U is a non-zero semigroup ideal of N, then, we get $(x, y) = 0$, for all $x, y \in N$.

This implies that $x + y = y + x$, for all $x, y \in N$.

Thus, (N,+) is abelain.

Lemma 4.2.6 (Majeed and Ahmed, 2011) Let N be a prime near-ring and U be a non-zero semigroup ideal of N. If U is commutative under multiplication and addition then N is a commutative ring.

Proof: For all $a, b \in U$.

$[a, b] = 0$. Taking ax instead of a and by instead of b, where $x, y \in N$.

We get $[ax, by] = 0$. Since U is a commutative semigroup ideal of N, then we have

$0 = axby - byax$

$= baxy - byax$

$= abxy - abyx$ (because U is commutative)

$= ab[x, y]$ for all $a, b \in U$, $x, y \in N$.

Thus ab[x, y] = 0 = U^2[x, y]

Since U is a Semigroup ideal of N.

U^2[rx, y] = {0}, for all r ∈ N

U^2r[x, y] = {0}, for all r ∈ N

We get U^2N[x, y] = {0}, for all x, y ∈ N.

Since N is a prime near-ring then either $U^2 = 0$ or [x, y] = 0 but U is a non-zero so $U^2 \neq 0$.

Then we get [x, y] = 0, for all x, y ∈ N.

Hence, N is a commutative ring.

Definition 4.2.3 (Boua *et al*., 2016) A near-ring is said to be 2-torsion free if 2x = 0, with x ∈ N, then x = 0.

4.3. Derivations on Prime Near Rings with Semigroup Ideals

In fact, in this part our discussion is focuses on derivations or non-zero derivations on prime near rings and semigroup ideals. We start this section by defining derivation on a prime near ring.

Definition 4.3.1 (Bell and Mason, 1987) An additive mapping d: N → N is defined to be a derivation on N if d(xy) =xd(y)+ d(x)y for all x, y ∈ N or equivalently, as (Wang, 1994) noted that d(xy) = d(x)y + xd(y) for all x, y ∈ N.

Example 4.3.1 (Bell and Mason, 1987) Let N be a zero-symmetric near-ring. If there exists a ∈ N such that Na ={0}, then the map d: N → N defined by d(x) = ax is a derivation on N.

Indeed, d is an additive map and d(xy) = axy = xay + axy = xd(y) + d(x)y.

By the same way, if a is distributive element in N and aN = {0}, then d(x) = xa is a derivation on N.

Lemma 4.3.1 (Bell and Mason, 1987) If d is a derivation on a near-ring N then for all $x, y, z \in N$ satisfies:

i. $(xd(y) + d(x)y)z = xd(y)z + d(x)yz$

ii. $(d(x)y + xd(y))z = d(x)yz + xd(y)z$

Proof:

i. We know that d(xy) = xd(y) + d(x)y, for all x, y ∈ N. Now

d((xy)z) = xyd(z) + d(xy)z

= xyd(z) + (xd(y) + d(x)y)z, for all x, y, z ∈ N, i.e.

d((xy)z) = xyd(z) + xd(y)z + d(x)yz, for all x, y, z ∈ N. (4.11)

Also, d(x(yz)) = xd(yz) + d(x)yz

= x(yd(z) + d(y)z) + d(x)yz, for all x, y, z ∈ N. i.e.

d(x(yz)) = xyd(z) + xd(y)z + d(x)yz, for ail x, y, z ∈ N. (4.12)

But we know that in a near ring N associative law holds,

i.e. (xy)z = x(yz), for all x, y, z ∈ N.

Then d((xy)z) = d(x(yz)), for all x, y, z ∈ N. Now from (4.11)

and (4.12), we get xd(y)z + d(x)yz = xd(y)z + d(x)yz, for all x, y, z ∈ N,

i.e. (xd(y) + d(x)y)z = xd(y)z + d(x)yz, for all x, y, z ∈N.

ii. We know that d(xy) = d(x)y + xd(y), for all x, y ∈ N.

Then d(x(yz)) = d(x)yz + xd(yz)

= d(x)yz + x(d(y)z + yd(z)), for all x, y, z ∈ N,

i.e. d(x(yz)) = d(x)yz + xd(y)z + xyd(z), for all x, y, z ∈ N. (4.13)

Now we take

d((xy)z) = d(xy)z + (xy)d{z)

= (d(x)y + xd(y))z + xyd(z), for all x, y, z ∈ N, i.e.

d((xy)z) = d(x)yz + xd(y)z + xyd(z) for all x, y, z ∈ N. (4.14)

Also we know that in a near ring associative law holds.

So we get x(yz) = (xy)z, for all x, y, z ∈ N.

This implies that d(x(yz)) = d((xy)z), for all x, y, z ∈ N. Now from

(4.13) and (4.14) we obtain d(x)yz + xd(y)z = d(x)yz + xd(y)z, for all x, y, z ∈ N.

Thus we have, (d(x)y + xd(y))z = d(x)yz + xd(y)z, for all x, y, z ∈ N.

The properties of Lemma4.3.1 in above i and ii are called partial right distributive properties.

Lemma 4.3.2 (Wang, 1994) Let d be a non-zero derivation of prime near-ring N and a be an element of N. If $ad(x) = 0$ for all $x \in N$, then either $a = 0$ or $d= 0$.

Proof: Suppose that $ad(x) = 0$ for all $x \in N$.

replacing x by xy. We have

$ad(xy) = 0 =ad(x)y + axd(y)$.

Then $axd(y) = 0$ for all $x, y \in N$

Know $aNd(y) = 0$. Since N is prime near ring and

since $d \neq 0$, such that $d(y) \neq 0$ for some $y \in N$,

which implies that $a = 0$.

Lemma 4.3.3 (Wang, 1994) Let d be an arbitrary additive mapping on N, Then $d(xy) = xd(y) + d(x)y$ if and only if $d(xy) = d(x)y+ xd(y)$ for all $x, y \in N$.

Proof: Suppose that $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$.

Since N satisfies left distributive law and $x(y+ y) = xy + xy$, we have

$d(x(y+ y)) =xd(y+ y) + d(x)(y+ y)$

$=x(d(y) + d(y)) + d(x)y+ d(x)y$

$=xd(y) + xd(y) + d(x)y+ d(x)y$

and

$d(xy+ xy) = d(xy) + d(xy)$

$= xd(y) + d(x)y+ xd(y) + d(x)y$.

Comparing these two equalities, we have

$xd(y) + d(x)y= d(x)y+ xd(y)$.

Hence $d(xy) = d(x)y + xd(y)$.

Lemma 4.3.4 (Bell, 1997) Let d be a derivation on near ring N and U be a semigroup ideal of N, suppose $u \in U$ is not a left zero divisor. If $[u, d(u)] = 0$, then $(x. u)$ is a constant for every $x \in U$

Proof: From $u(u + x) = u^2 + ux$.

Apply d for both sided we have

$d(u(u + x)) = d(u^2 + ux)$.From this equation, we have

$ud(u + x) + d(u)(u + x) = ud(u) + d(u)u + ud(x) + d(u)x$.

Which reduces to $ud(x) + d(u)u = d(u)u + ud(x)$, for $u, x \in U$.

By using the hypothesis $[u, d(u)] = 0$, this equation is expressible as

$u(d(x) + d(u) - d(x) - d(u)) = 0 = u(d(x, u))$. Since u is not a left zero divisor.

We get $d((x, u)) = 0$. Thus, (x, u) is a constant for every $x \in U$.

Lemma 4.3.5 (Bell, 1997) Let d be a non-zero derivation on a prime near ring N and U be a non-zero semigroup ideal of N .Then $xd(U) = 0$ implies $x = 0$ and $d(U)x = 0$ implies $x = 0$.

where $x \in N$.

Proof: Let $xd(U) = 0$. For any $r \in N$, $s \in U$.

Then $xd(sr) = 0$ for $x, r \in N$ and $s \in U$.

Thus, $xsd(r) + xd(s)r = 0$, the second term in this equation equal zero by the hypothesis, we get $xsd(r) = 0$ for $x, r \in N$ and $s \in U$

hence $xUd(r) = 0$. Since U is a semigroup ideal of N,

We get $xUNd(r) = 0$. Since N is a prime near ring and U is a non-zero semigroup ideal, d is a non-zero derivation of N, we get $x = 0$. By similar way, we can show that if $d(U)x = 0$, for all $x \in N$ implies that $x = 0$.

Corollary 4.3.1 (Ashraf and Ali, 2008) Let N be a 2-torsion free prime near-ring. Suppose that d is a non-zero derivation on N such that d([x, y]) = 0, for all x, y ∈ N or d(x o y) = 0, for all x, y ∈ N. Then N is a commutative ring.

Lemma 4.3.6 (Bell, 1997) If N is a prime near ring and Z(N) contains a non –zero semigroup left ideal or a semigroup right ideal, then N is a commutative ring.

Lemma 4.3.7 (Deng *et al.,*1998) Let N is an n! torsion free prime near ring admitting a derivation d such that d(N) ⊆ Z(N), then either d(Z(N)) = 0 or N is a commutative ring.

Theorem 4.3.1 (Bell, 1997) Let N be a prime near ring and let U he a. non-zero semigroup right ideal (or a non-zero semigroup left ideal) of N. If N admits a non-zero derivation d for which d(U) ⊆ Z(N), then N is a commutative ring.

Theorem 4.3.2 (Bell, 1997) Let N be a near ring with no non- zero divisor of zero, and U be a non-zero semigroup right ideal of N. If N admits a non-zero derivation d such that d(xy) = d(yx) for all x, y ∈ U, then N is a. commutative ring.

Theorem 4.3.3 (Boua and Oukhtite, 2011) Let N be prime near-ring. If N admits anon-zero derivation d satisfying d([x, y]) = [x, y] for all x, y ∈ N. Then d is commuting (resp. centralizing derivation on N).

4.4. Generalized Derivations on Prime Near Ring with Semigroup Ideals

The main aim of this section is to describe generalized derivations in prime near rings. We start from some notes on generalized derivation given as follows

Note (Boua et al., 2015)

i. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting.
ii. Every derivation on near ring N is a generalized derivation.

Familiar examples of generalized derivations are derivations and generalized inner derivations and later includes left multiplier i.e. an additive mapping f: N → N satisfying

f(xy) = f(x)y for all x, y ∈ N.

Definition 4.4.1 (Golbasi, 2006) Let N be a near ring and d a derivation of N.

An additive mapping f: N → N is said to be a right generalized derivation of N

associated with d if

I. $f(xy) = f(x)y + xd(y)$ for all x, y ∈ N: and f is said to be a left generalized derivation of N associated with d if

II. $f(xy) = d(x)y + xf(y)$ for all x, y ∈ N

Finally, f is said to be a generalized derivation of N associated with d if it is both a left and right generalized derivation of N.

Proposition 4.4.1 (Kamal and Al-Shaalan, 2013) Let N be a prime near-ring with f a generalized derivation associated with a derivation d. If f = 0, then d = 0.

Proof: From $f(xy) = f(x)y + xd(y)$ for all x, y ∈ N and f = 0,

We have $xd(y) = 0$ for all x, y ∈ N.

By Lemma 4.2.2, we have d = 0.

Lemma 4.4.1 (Ali et al., 2013) Let N be a prime near-ring with a non-zero semigroup ideal U and a generalized derivation f such that $f(U) = \{0\}$. Then f = 0

Proof: Let N be a near-ring. for all u ∈ U and for all x ∈ N

We have $0 = f(ux) = f(u)x + ud(x) = ud(x)$.

Since $U \neq \{0\}$, we get d = 0 by Lemma 4.2.2.

It follows that $f(xy) = f(x)y$ for all x, y ∈ N.

Replacing y by u ∈ U,

we conclude that $0 = f(xu) = f(x)u$.

Using Lemma4.2.2 again, we get that f = 0.

Lemma 4.4.2 (Bell and Mason, 1987) Let N be a prime near ring.

i. If $z \in Z(N) \setminus \{0\}$, then z is not a zero divisor in N.
ii. If $Z(N) \setminus \{0\}$ contains an element z such that $z + z \in Z(N)$, then (N, +) is abelian.

Proof: i. If $z \in Z(N)\setminus\{0\}$ and $zx = 0$, then $zNx = \{0\}$,

Since N is prime, so it follows that x = 0.

Hence z is not a zero divisor.

ii. Let $z \in Z(N) \setminus \{0\}$ be an element such that $z + z \in Z(N)$,

and let $x, y \in N$.
since $z + z$ is distributive,
we get, $(x + y)(z + z) = x(z + z) + y(z + z)$
$= xz + xz + yz + yz$
$= z(x + x + y + y)$
On the other hand,
$(x + y)(z + z) = (x + y)z + (x + y)z$
$= xz + yz + xz + yz$
$= z(x + y + x + y)$

Thus, $x + x + y + y = x + y + x + y$

This shows $x + y = y + x$

Hence $(N, +)$ is abelian.

Theorem 4.4.1 (Ali *et al.,* 2013) Let N be a prime near ring and U be a non-zero semigroup ideal of N. If f is a non-zero right generalized derivation of N with associated derivation d and

$f(x)y = xf(y)$, for all $x, y \in U$, then $d = 0$.

Proof: We are given that $f(x)y = xf(y)$, for all $x, y \in U$.

Substituting yz for y, we get $f(x)yz = xf(yz)$

$= x\,(f(y)z + yd(z))$ for all $x, y, z \in U$.

It follows that $xyd\,(z) = 0$ for all $x, y, z \in U$,

that is, $xUd(z) = \{0\}$ for all $x, z \in U$.

By Lemma4.2.3, $d\,(z) = 0$, and hence $d = 0$

Theorem 4.4.2 (Bell and Mason, 1987) Let N be a prime near-ring, and let f be a generalized derivation on N with associated derivation d. If $f^2 = 0$, then $d^3 = 0$.

Moreover, if N is 2-torsion free, then $d\,(Z\,(N)) = \{0\}$.

Proof: We have $f^2\,(xy) = f\,(f(x)\,y + x\,d(y))$

$= f\,(x)d(y) + f(x)d(y) + xd^2(y)$

$= 0, \ \forall x, y \in N$ (4.15)

Applying f to above equation (4.15) gives

$f(x)\, d^2(y) + f(x)\, d^2(y) + f(x)d^2(y) + xd^3(y) = 0, \ \forall x, y \in N.$ (4.16)

Substituting d(y) for y in equation (5.11)

gives $f(x)d^2(y) + f(x)d^2(y) + xd^3(y) = 0;$ (4.17)

Therefore, by equations (4.16) and (4.17),

$f(x)d^2(y) = 0 \quad \forall x, y \in N.$ (4.18)

Know it follows from equation (4.17)

that $xd^3(y) = 0$ for all $x, y \in N$; and since N is prime, $d^3 = 0$.

Suppose know that N is 2-torsion-free and that $d(Z(N)) \neq \{0\}$,

and let $z \in Z(N)$ be such that $d(z) \neq 0$.

Then if $x, y \in N$ and $f(N)x = \{0\}$,

then $f(yz)x = f(y)zx + yd(z)x = 0 = yd(z)x$;

and since N is prime and d(z) is not a zero divisor, $x = 0$.

Know it follows from equation (4.18) that $d^2 = 0$ and hence $d = 0$.

But this contradicts our assumption that $d(Z(N)) \neq \{0\}$,

Hence $d(Z(N)) = \{0\}$.

Theorem 4.4.3 (Golbasi, 2006) Let (f, d) be a generalized derivation of a prime near-ring N such that $d(Z(N)) \neq \{0\}$, and $a \in N$. If $[f(x), a] = 0$ for all $x \in N$, then $a \in Z(N)$.

Proof: Since $d(Z(N)) \neq \{0\}$, there exists $c \in Z(N)$ such that $d(c) \neq 0$.

Furthermore, as d is a derivation, it is clear that $d(c) \in Z(N)$.

Replacing x by cx

We have $f(cx)a = af(cx)$ i.e., $D(c)xa + cf(x)a = ad(c)x + acf(x)$.

Since $c \in Z(N)$ and $d(c) \in Z(N)$,

We get $D(c)N\backslash[x, a] = 0$ for all $a \in N$.

By the primeness of N and $0 \neq d(c) \in Z(N)$,

We obtain that $a \in Z(N)$.

Theorem 4.4.4 (Ashraf and Siddeeque, 2015) Let N be prime near ring without non-zero divisors of zero, and U a non-zero semigroup right ideal of N. If N admits a left generalized derivation (f, d) such that $f([x, y]) = 0$ for all $x, y \in U$, then N is a commutative ring.

Proof: Assume that $f([x, y]) = 0$ for all $x, y \in U$.

Putting xy in place of y, obtaining

$$f([x, xy]) = f(x[x, y])$$
$$= d(x)[x, y] + xf([x, y])$$
$$= 0.$$

Since the second term is zero.

It is clear that $d(x)[x, y] = 0$ for all $x, y \in U$.

Thus by hypothesis for each $x \in U$, either

$d(x) = 0$ or x centralizes U.

We see that either

$d(x) = 0$ or $x \in Z(N)$.

If $x \in Z(N)$ then $xr = rx$ for all $r \in N$,

This gives us,

$d(x)r + xd(r) = rd(x) + d(r)x$ for all $r \in N$.

Previous relation implies that $d(x) \in Z(N)$.

Therefore we conclude that $d(U) \subseteq Z(N)$.

Thus N is commutative.

Theorem 4.4.5 (Ali *et al*., 2013) Let N be a prime near ring and U a non-zero semigroup ideal of N. If f is a non-zero left generalized derivation with associated non-zero derivation d such that

$f([x, y]) = [x, y]$ for all $x, y \in U$ or

$f([x, y]) = -[x, y]$ for all $x, y \in U$, then N is a commutative ring.

Proof: We assume $f([x, y]) = [x, y]$ for all $x, y \in U$ (4.19)

The other condition is treated in the same fashion.

Replacing y by xy in (4.19)

We obtain $f(x[x, y]) = x[x, y]$ for all $x, y \in U$. (4.20)

Now $f(x[x, y]) = d(x)[x, y] + x\, f([x, y])$; and using (4.19), We find that

$f(x[x, y]) = d(x)[x, y] + xf[x, y]$ for all $x, y \in U$. (4.21)

Comparing (4.20) and (4.21),

we get $d(x)[x, y] = 0$ for all $x, y \in U$, (4.22)

Know we obtain that

$d(x)xy - d(x)yx = 0$ for all $x, y \in U$. (4.22)

Replacing y by yz in (4.22),

We get $d(x)xyz - d(x)yzx = 0$ for all $x, y \in U, z \in N$;

And using (4.22),

we get $d(x)yxz - d(x)yzx = 0$ for all $x, y \in U, z \in N$,

This can be rewritten as $d(x)U[x, z] = 0$ for all $x \in U, z \in N$.

Therefore, for each $x \in U$, either $x \in Z(N)$ or $d(x) \in Z(N)$.

Thus $d(U) \subseteq Z(N)$.

Thus N is a commutative.

4.5. Generalized Derivations on Prime Near rings Acting as Homomorphism or Anti-Homomorphism

(Bell and Kappe, 1989) initiated the study of derivations which act as homomorphisms or as anti-homomorphisms on a ring. They also proved that if R is a prime ring and d is a derivation of R which acts as a homomorphism or as an anti -homomorphism, then $d = 0$.

Theorem 4.5.1 (Golbasi, 2006) Let (f, d) be a non-zero generalized derivation of N. If f acts as a homomorphism on N, then f is the identity map.

Proof: Assume that f acts as a homomorphism on N. Then one obtains

$f(xy) = f(x)f(y)$

$= d(x)y + xf(y)$ for all x, y $\in$ N. (4.23)

Replacing y by yz in (4.23), we have

$f(x)f(yz) = d(x)yz + xf(yz)$.

Since (f, d) be a non-zero generalized derivation on N and f acts as a homomorphism on N, we deduce that

$f(xy)f(z) = d(x)yz + xd(y)z + xyf(z)$.

By Lemma 4.3.1(ii), we get

$d(x)yf(z) + xf(y)f(z) = d(x)yz + xd(y)z + xyf(z)$, and so

$d(x)yf(z) + xf(yz) = d(x)yz + xd(y)z + xyf(z)$.

That is,

$d(x)yf(z) + xd(y)z + xyf(z) = d(x)yz + xd(y)z + xyf(z)$.

Hence, we deduce that $d(x)y(f(z) - z) = 0$ for all x, y, z $\in$ N.

Because N is prime and $d \neq 0$,

We have $f(z) = z$ for all z $\in$ N.

Thus, f is the identity map.

Theorem 4.5.2 (Ali *et al.*, 2013) Let N be a prime near ring and U be a nonzero semigroup ideal of N. Let f be a non -zero generalized derivation on N with associated derivation d. If f acts as a homomorphism on U, then f is the identity map on N and $d = 0$.

Proof: By the hypothesis

$f(xy) = d(x)y + xf(y) = f(x)f(y)$ for all $x, y \in U$. (4.24)

Replacing y by yz in the above relation, we get

$f(xyz) = d(x)yz + xf(yz)$ for all $x, y, z \in U$, (4.25)

or

$f(xy)f(z) = d(x)yz + x(d(y)z + yf(z))$ for all $x, y, z \in U$. (4.26)

This implies that

$(d(x)y + x\mathrm{f}(y))\,\mathrm{f}(z) = d(x)yz + xd(y)z + xy\mathrm{f}(z)$ for all $x, y, z \in U$ (4.27)

Using Lemma4.3.1 (ii), we get

$d(x)y\mathrm{f}(z) + x\mathrm{f}(y)\,\mathrm{f}(z) = d(x)yz + xd(y)z + xy\mathrm{f}(z)$ for all $x, y, z \in U$, (4.28)

or

$d(x)y\mathrm{f}(z) + x\mathrm{f}(yz) = d(x)yz + xd(y)z + xy\mathrm{f}(z)$ for all $x, y, z \in U$. (4.29)

This implies that

$d(x)y\mathrm{f}(z) + xd(y)z + xy\mathrm{f}(z) = d(x)yz + xd(y)z + xy\mathrm{f}(z)$ for all $x, y, z \in U$. (4.30)

That is,

$d(x)y\mathrm{f}(z) = d(x)yz$ for all $x, y, z \in U$. (4.31)

Therefore

$d(x)y(\mathrm{f}(z) - z) = 0$ for all $x, y, z \in U$, (4.32)

Which implies that $d(x)U(\mathrm{f}(z) - z) = \{0\}$ for all $x, z \in U$. (4.33)

It follows by Lemma4.2.4 that either $d(U) = 0$ or

$\mathrm{f}(z) = z$ for all $z \in U$.

In fact, as we now show, both of these conditions hold.

Suppose that $\mathrm{f}(u) = u$ for all $u \in U$.

Then for all $u \in U$ and $x \in N$

$\mathrm{f}(xu) = xu$

$= d(x)u + x\mathrm{f}(u)$

$= d(x) + xu$; hence

$d(x)U = \{0\}$ for all $x \in N$, and thus $d = 0$. On the other hand,

Suppose that$d(U) = \{0\}$,so that $d = 0$.

Then for all $x, y \in U$, $f(xy) = f(x)y = f(x)f(y)$, so that

$f(x)(y - f(y)) = 0$.

Replacing$y$ by zy, $z \in N$, and noting that

$f(zy) = zf(y)$, we see that $f(x)N(y - f(y)) = \{0\}$ for all $x, y \in U$.

Therefore, $f(U) = \{0\}$ or f is the identity on U .But $f(U) = \{0\}$ contradicts Lemma4.4.1,

so f is the identity on U.

We now know that f is the identity on U and

$f(xy) = xf(y)$ for all $x, y \in N$.

Consequently,

$f(ux) = ux = uf(x)$ for all $u \in U$ and $x \in N$, so that $U(x - f(x)) = \{0\}$ for all $x \in N$.

It follows that f is the identity on N.

Theorem 4.5.3 (Golbasi, 2006) Let (f, d) be a non-zero generalized derivation of N. If f acts as an anti-homomorphism on N, then f is the identity map.

Proof: By the hypothesis, we have

$f(xy) = f(y)f(x) = d(x)y + xf(y)$ for all $x, y \in N$. (4.34)

Replacing y by xy in the equation (4.24), we obtain

$f(xy)f(x) = d(x)xy + xf(xy)$.

Since (f, d) is a generalized derivation and f acts as an anti-homomorphism on N,

We get $(d(x)y + xf(y))f(x) = d(x)xy + xf(y)f(x)$.

By Lemma 4.3.1(ii), we conclude that

$d(x)yf(x) + xf(y)f(x) = d(x)xy + xf(y)f(x)$, and so

$d(x)yf(x) = d(x)xy$ for all $x, y \in N$. (4.35)

Replacing y by yz in equation (4.35) and using equation (4.35),

We have $d(x)N[f(x),z] = 0$ for all $x, z \in N$.

Hence we obtain the following alternatives: d(x) = 0 or f(x) ∈ Z, for all x ∈ N.

Because of N is prime and since d ≠ 0,

We have f(x) = x for all x ∈ N

Thus, f is the identity map.

5. SOME RESULTS ON GENERALIZED REVERSE DERIVATIONS ON PRIME NEAR RINGS WITH SEMIGROUP IDEALS

Our main objective in this chapter is to describe some recent work in the area of the commutativity of prime near rings with generalized reverse derivations. The results presented in this chapter are based on the work of Reddy *et al.* (2015). Section 5.1 begins with the definition of reverse derivation in prime near ring and the results presented in this section are the results obtained earlier in the prime near rings in the setting of semigroup ideals with derivations and generalized derivations. The discussions continue in sections 5.2 up 5.4. Finally Section 5.5 is devoted to describe commutativity of prime near rings in the setting of semigroup ideals. In this chapter, also the researcher investigated Theorem 5.3.4 – 5.3.7 and Theorem 5.4.3, and Theorem 5.4.4 and also lastly theorem 5.5.5 and 5.5.6.

5.1. Reverse Derivations on Prime Near Rings

The main focus of this section is to discuss prime near rings with reverse derivations. The results presented in this section are more based on the work of Reddy et al. (2015).

Definition 5.1.1 (Narayana *et al.,* 2020) An additive map d: N → N is a reverse derivation if d(xy) = d(y)x + yd(x) for all x, y in N.

Lemma 5.1.1 (Narayana *et al.,* 2020) If N is a near ring and d is a reverse derivation of N, then a(d(y)x + yd(x)) = ad(y)x + ayd(x), for all a, y, x ∈ N.

Proof: We know that d(xy) = d(y)x + yd(x) for all x, y ∈ N. Now suppose that

d(a(xy)) = xyd(a) + ad(xy) for all a, x, y ∈ N. by lemma 4.3.1

= xyd(a) + a(d(y)x + yd(x)) for all a, x, y ∈ N by definition

= xyd(a) + ad(y)x + ayd(x) for all a, x, y ∈ N (5.07)

Also, d(ay)x = xd(ay) + ayd(x) for all a, x, y ∈ N by lemma 4.3.1

$= x(d(y)a + yd(a)) + ayd(x)$ for all $a, x, y \in N$.

$= xd(y)a + xyd(a) + ayd(x)$ for all $a, x, y \in N$. (5.08)

But we know that in near ring N associative law holds,

i. e. $a(xy) = (ay)x$, for all $a, x, y \in N$.

Then $d(a(xy)) = d(ay)x$, for all $a, x, y \in N$. Now from equation (5.07) and equation (5.08),

We get $ad(y)x + ayd(x) = ad(y)x + ayd(x)$ for all $a, x, y \in N$, i.e.

$a(d(y)x + yd(x)) = ad(y)x + ayd(x)$ for all $a, x, y \in N$.

Lemma 5.1.2 (Reddy *et al.,* 2015)

a) Let d be a non-zero reverse derivation on prime near ring N. Then $xd(N) = \{0\}$ implies

$x = 0$ and $d(N)x = \{0\}$ implies $x = 0$

b) If N is 2- torsion free and d is a reverse derivation on prime near ring N such that $d^2 = 0$, then $d = 0$.

Proof:

a) Let $xd(N) = \{0\}$, and let r, s be arbitrary elements of N. Then

$0 = xd(rs)$

$= x(d(s)r+sd(r))$

$= xd(s)r+xsd(r)$

$= xsd(r)$

Thus, $xNd(N) = \{0\}$ and since N is a prime near ring, we have either $x = 0$(or)

$d(N) = 0$. Since d is a non-zero reverse derivation on N, we have $x = 0$.

i.e., since $xd(N) = \{0\} \rightarrow x = 0$. A similar argument works if $d(N)x = \{0\} \rightarrow x = 0$.

b) For arbitrary $x, y \in N$, we have

$0 = d^2(xy)$

$= d(d(xy))$

$= d(d(y)x + yd(x))$

$= d(d(y)x) + d(yd(x))$ by left distributive law

$= d(x)d(y) + xd^2(y) + d^2(x)y + d(x)d(y)$

$= 2d(x)d(y)$

This implies that $2d(x)d(N) = \{0\}$, as $y \in N$.

Since N is 2-torsion-free, it follows that $d(x)d(N) = \{0\}$, for each $x \in N$.

Thus we get $d = 0$.

Lemma 5.1.3 (Reddy *et al.,* 2015) Let d be a reverse derivation on N, and suppose $u \in N$ is not a right zero divisor. If $[u, d(u)] = 0$, then (x, u) is a constant for every $x \in N$

Proof: From $u(u + x) = u^2 + ux$, we obtain that

⇨ $(d(u + x))u + (u + x)d(u) = d(u^2) + d(ux) = d(u \cdot u) + d(ux)$

⇨ $d(x)u + d(u)u = d(u)u + d(x)u$

⇨ $(d(x) + d(u) - d(x) - d(u))u = 0 = d((x,u))$

⇨ $d((x, u)) = 0$

Thus (x, u) is a constant for every $x \in N$.

Theorem 5.1.1 (Reddy *et al.,* 2015) Let d be an arbitrary reverse derivation of a right near ring N. Then N satisfies the following partial distributive law.

$z(d(y)x + yd(x)) = zd(y)x + zyd(x)$ for all $x, y, z \in N$.

Proof: We know that $d(xy) = d(y)x + yd(x)$ for all $x, y \in N$.

Now $d(x(yz)) = d(yz)x + yz\,d(x)$ for all $x, y, z \in N$.

$= (d(z)y + zd(y))x + yzd(x)$ by assumption

$= d(z)yx + zd(y)x + yzd(x)$ by right distributive law

That means $d(x(yz)) = d(z)yx + zd(y)x + yzd(x)$ for all $x, y, z \in N$. (5.09)

Also, $d((xy)z) = d(z)xy + zd(xy)$ for all $x, y, z \in N$.

$= d(z)xy + z(d(y)x + yd(x))$

$= d(z)xy + zd(y)x + zyd(x)$

This shows $d((xy)z) = d(z)xy + zd(y)x + zyd(x)$. (5.10)

But we know that in a near ring N is associative law holds that means

$x(yz) = (xy)z$ for all $x, y, z \in N$.

Now $d(x(yz)) = d(xy)z)$ for all $x, y, z \in N$. So, from equation (5.09) and (5.10)

we get $z(d(y)x + yd(x)) = zd(y)x + zyd(x)$ for all $x, y, z \in N$.

5.1.1. Reverse Derivations on Prime Near Rings with Semigroup Ideals

Reddy *et al.* (2015) proved result in near ring know we have the following result in semigroup ideals.

Lemma 5.1.1.1 Let d be a reverse derivation on prime near ring N and U be a non-zero semigroup ideal of N. Suppose $u \in U$ is not a right zero divisor. If $[u, d(u)] = 0$, then (x, u) is a constant for every $x \in U$.

Proof: From $u(u + x) = u^2 + ux$, we obtain that

$\Rightarrow (d(u + x))u + (u + x)d(u) = d(u^2) + d(ux) = d(u \cdot u) + d(ux)$

$\Rightarrow d(x)u + d(u)u = d(u)u + d(x)u$

$\Rightarrow (d(x) + d(u) - d(x) - d(u))u = 0$

$= d((x,u))$

$\Rightarrow d((x, u)) = 0$

Thus (x, u) is a constant for every $x \in U$.

Lemma 5.1.1.2 Let N be a prime near ring admitting a non-zero reverse derivation d and U be a non-zero semigroup ideal of N. Then the following hold:

i. If $x, y \in N$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.
ii. If $x, y \in N$ and $d(U)x = \{0\}$, then $x = 0$.
iii. If $x, y \in N$ and $xd(U) = \{0\}$, then $x = 0$.

Proof: To obtain (i) suppose that $xUy = \{0\}$. Then $xUNy = \{0\}$, Hence $y = 0$ or

$xU = \{0\}$. In the latter case $x = 0$ by Lemma 4.2.4.
For (ii), suppose $d(U)x = \{0\}$. By lemma 4.3.5 we see that for
all $u \in U$ and $y \in N$. $0 = d(yu)x = (ud(y) + d(u)y)x = ud(y)x + d(u)yx$. Hence
$ud(y)x = \{0\}$, for all $y \in N$ and $x = 0$ by (i).

Lemma 5.1.1.3 Let N be a prime near ring and U be a non-zero semigroup ideal of N. If d is a non-zero reverse derivation on N, then $d(U) \neq \{0\}$.

Proof: let U be a non-zero semigroup right ideal.

Suppose that $d(U) = \{0\}$. Then for all $u \in U$ and $x \in N$

$0 = d(ux) = xd(u) + ud(x) = ud(x)$, that is

$Ud(x) = 0$, for all $u \in U$. This implies that

$Ud(x) = \{0\}$. Hence $d = 0$, since U is a non-zero semigroup ideal of N.

But this contradicts our given assumption that d is a non-zero reverse derivation.

Thus $d(U) \neq \{0\}$.

Initiated by Reddy *et al.* (2015) above results for near rings..We prove the following results for prime near rings in the setting of semigroup ideals.

Theorem 5.1.1.1 Let d be an arbitrary reverse derivation of a right near ring N on a semigroup ideal U of N. Then N satisfies the following partial distributive law.

$z(d(y)x + yd(x)) = zd(y)x + zyd(x)$ for all $x, y, z \in U$.

Proof: We know that $d(xy) = d(y)x + yd(x)$ for all $x, y \in U$.

Now $d(x\,(yz)) = d(yz)x + yz\,d(x)$ for all $x, y, z \in U$.

$= (d(z)y + zd(y))\,x + yzd(x)$

$= d(z)yx + zd(y)x + yzd(x)$.

That means $d(x\,(yz)) = d(z)yx + zd(y)x + yzd(x)$ for all $x, y, z \in U$. (5.11)

Also, $d((xy)z) = d(z)xy + zd(xy)$ for all $x, y, z \in U$.

$= d(z)xy + z(d(y)x + yd(x))$

$= d(z)xy + zd(y)x + zyd(x)$

This shows $d((xy)z) = d(z)xy + zd(y)x + zyd(x)$. (5.12)

But we know that in a near ring N is associative law holds that means

$x(yz) = (xy)z$ for all $x, y, z \in U$.

Now $d(x(yz)) = d(xy)z)$ for all $x, y, z \in U$. So, from equation (5.11) and (5.12)

We get $z(d(y)x + yd(x)) = zd(y)x + zyd(x)$ for all $x, y, z \in U$.

5.2. Generalized Reverse Derivations on Prime Near Rings with Semigroup Ideals

Definition 5.2.1 (Reddy *et al.,* 2015) An additive mapping defined that $f: N \rightarrow N$ is a right (resp., left) generalized reverse derivation if there exists a derivation d from N to N such that $f(xy) = f(y)x + yd(x)$ (resp., $f(xy) = d(y)x + yf(x)$), for all $x, y \in N$,
finally, f is a generalized reverse derivation of N associated with d if it is both right and left generalized reverse derivation of N.

Example 5.2.1 Let S be any near ring and let $N = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \middle| \ a, b, c \in S \right\}$

Then N is near ring under matrix addition and matrix multiplication.

Define the mapping $F: N \rightarrow N$ and $D: N \rightarrow N$ such that $F \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\text{and } D\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & a-c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then N admits a non – zero generalized reverse derivation f associated with a non – zero derivation D.

Lemma 5.2.1 (Reddy et al., 2015)

(i) Let f be a right generalized reverse derivation of a near ring N associated with d. Then $f(xy) = yd(x) + f(y)x$, for all $x , y \in N$.

(ii) Let f be a left generalized reverse derivation of a near ring N associated with d. Then $f(xy) = yf(x) + d(y)x$, for all $x , y \in N$.

Proof

(i) We suppose that $f(xy) = f(y)x + yd(x)$ for all $x ,y \in N$.

From $x(y + y) = xy + xy$ and N satisfies left distributive law

$$\Rightarrow f(x(y+y)) = (f(y+y))x + (y+y)d(x)$$

$$= f(y)x + f(y)x + yd(x) + yd(x) \quad (5.13)$$

and $f(xy + xy) = f(xy) + f(xy)$

$$= f(y)x + yd(x) + f(y)x + yd(x) \quad (5.14)$$

By comparing above 5.13 and 5.14 equations, we have

$$f(y)x + yd(x) = yd(x) + f(y)x.$$

Hence $f(xy) = yd(x) + f(y)x$

(ii) We suppose that $f(xy) = d(y)x + yf(x)$ for all $x , y \in N$

From $(x + x)y = xy + xy$ and N satisfies right distributive law

$$\Rightarrow f((x+x)y) = d(y)(x+x) + y f(x+x)$$

$$= d(y)x + d(y)x + y f(x) + y f(x) \quad (5.15)$$

and $f(xy + xy) = f(xy) + f(xy)$

$$= d(y)x + y f(x) + d(y)x + y f(x) \quad (5.16)$$

By comparing 5.15 and 5.16 equalities, we have

$$\Rightarrow d(y)x + y f(x) = y f(x) + d(y)x$$

Hence $f(xy) = yf(x) + d(y)x$

Lemma 5.2.2 (Reddy *et al.*, 2015)

(i) Let f be a right generalized reverse derivation of the near ring N associated with d. Then $(f(y)x + yd(x))z = f(y)xz + yd(x)z$ for all $x, y, z \in N$.

(ii) Let f be a left generalized reverse derivation of the near ring N associated with d. Then

$(d(y)x + yf(x))z = d(y)xz + yf(x)z$, for all $x, y, z \in N$

Lemma 5.2.3 (Reddy et al., 2015) Let N be a prime near ring, f a non-zero generalized reverse derivation of N, associated with the non-zero derivation d, and $a \in N$.

(i) If $af(N) = 0$, then $a = 0$.

(ii) If $f(N)a = 0$, then $a = 0$.

Proof: (i) We suppose that $af(N) = \{0\}$, for $a \in N$. Then for all $b, r \in N$,

$$0 = af(br)$$
$$= a(f(r)b + rd(b))$$
$$= af(r)b + ard(b)$$
$$0 = ard(b)$$

Hence $aNd(b) = \{0\}$, for all $b \in N$.

Since by hypothesis, d is a non-zero derivation on N, then we must have $a = 0$.

(ii) We suppose that $f(N)a = 0$, for all $a \in N$.

Then for all $b, r \in N$,

$$0 = f(rb)a$$
$$= (d(b)r + bf(r))a$$
$$= d(b)ra + bf(r)a$$
$$= d(b)ra$$

Hence $d(b)Na = \{0\}$, for all $b \in N$.

Since d is a non-zero derivation on N, we have $a = 0$.

Lemma 5.2.4 (Reddy et al., 2015) Let d be a non-zero reverse derivation of a prime near ring N and $a \in N$ such that $ad(x) = 0((\text{or})d(x)a = 0)$, for all $a \in N$. Then $a = 0$

Proof: We suppose that $ad(x) = 0$, for all $x \in N$.

By replacing x by yx, we have

⇨ $ad(yx) = 0$

⇨ $a(d(x)y + xd(y)) = 0$

⇨ $ad(x)y + axd(y) = 0$

⇨ $axd(y) = 0$, for all $x, y \in N$.

Therefore $aNd(y) = 0$ By primeness of N, we have either $a = 0$ (or) $d = 0$.

Since d is a non-zero derivation on N, then $a = 0$.

Initiated by above results we prove the following lemmas for semigroup ideals.

Lemma 5.2.5

(i) Let f be a right generalized reverse derivation of a near ring N associated with d and U be a non - zero semigroup ideal of N . Then $f(xy) = yd(x) + f(y)x$, for all $x , y \in U$.

(ii) Let f be a left generalized reverse derivation of a near ring N associated with d and U be a non - zero semigroup ideal of N. Then $f(xy) = yf(x) + d(y)x$, for all $x , y \in U$.

Proof:

(i) We suppose that $f(xy) = f(y)x + yd(x)$ for all $x ,y \in U$.

From $x(y + y) = xy + xy$ and N satisfies left distributive law

⇨ $f (x(y + y)) = (f (y + y))x + (y + y)d(x)$

$$= f (y)x + f (y)x + yd(x) + yd(x) \quad (5.17)$$

and

$$f (xy + xy) = f(xy) + f(xy) = f(y)x + yd(x) + f(y)x + yd(x) \quad (5.18)$$

By comparing above 5.17 and 5.18 equations , we have

$f (y)x + yd(x) = yd(x) + f (y)x$.

Hence $f (xy) = yd (x) + f (y) x$

(ii) We suppose that $f (xy) = d(y)x + yf(x)$ for all $x , y \in U$.

From $(x + x)y = xy + xy$ and N satisfies right distributive law .

⇨ $f ((x + x)y) = d(y)(x + x) + y f (x + x)$

$$= d (y)x + d (y)x + y f (x) + y f (x) \quad (5.19)$$

and

$$f (xy + xy) = f (xy) + f (xy) = d (y)x +y f (x) + d (y)x + y f(x) \quad (5.20)$$

By comparing 5.19 and 5.20 equalities, we have

$d (y)x + y f (x) = y f (x) + d (y)x$

Hence $f (xy) = yf (x) + d (y)x$

Lemma 5.2.6

(i) Let f be a right generalized reverse derivation of the near ring N associated with d and U be a non - zero semigroup ideal of N.

Then $(f(y)x + yd(x))z = f(y)xz + yd(x)z$ for all $x, y, z \in U$.

(ii) Let f be a left generalized reverse derivation of the near ring N associated with d and U be a non - zero semigroup ideal of N.

Then $(d(y)x + yf(x))z = d(y)xz + yf(x)z$, for all $x, y, z \in U$.

Proof:

(i) We know that $f(xy) = f(y)x + yd(x)$ for all $x, y, z \in U$.

Now $f(x(zy)) = f(zy)x + zyd(x)$ for all $x, y, z \in U$

$$= (f(y)z + yd(z))x + zyd(x))$$

$$= f(y)zx + yd(z)x + zyd(x)$$

$$= yd(z)x + f(y)zx + zyd(x) \qquad (5.21)$$

and $f((xz)y) = f(y)xz + yd(xz)$ for all $x, y, z \in U$.

$$= f(y)xz + y(d(z)x + Zd(x))$$

$$= f(y)xz + y(d(z)x + d(x)z)$$

$$= f(y)xz + yd(z)x + yd(x)z$$

$$= yd(z)x + f(y)xz + yd(x)z \qquad (5.22)$$

Since $f(x(zy)) = f((xz)y)$ for all $x, y, z \in U$. f rom equations 5.21 and 5.22.

We get $f(y)xz + yd(x)z = f(y)xz + yd(x)z$ for all $x, y, z \in U$.

Thus $(f(y)x + yd(x))z = f(y)xz + yd(x)z$, for all $x, y, z \in U$.

(ii) We know that $f(xy) = d(y)x + yf(x)$ for all $x, y, z \in U$.

Now $f(x(zy)) = d(yz)x + zyf(x)$ for all $x, y, z \in U$.

$$= (d(z)y + zd(y))x + zyf(x)$$

$$= (yd(z) + zd(y))x + zyf(x)$$

$$= yd(z)x + zd(y)x + zyf(x) \qquad (5.23)$$

and $f((xz)y) = d(y)xz + yf(xz)$ for all $x, y, z \in U$

$$. = d(y)xz + y(d(z)x + zf(x))$$

$$= d(y)xz + yd(z)x + yf(x)z$$

$$= yd(z)x + d(y)xz + yf(x)z \qquad (5.24)$$

Since $f(x(zy)) = f((xz)y)$ for all $x, y, z \in U$.

From equations 5.23 and 5.24 we get

$d(y)xz + y f(x)z = d(y)xz + y f(x)z$, for all $x , y, z \in U$.

Thus $(d(y)x + y f(x))z = d(y)xz + yf(x)z$, for all $x , y, z \in U$.

Lemma 5.2.7 Let N be a prime near ring and U be a non-zero semigroup ideal of N. If f a non-zero generalized reverse derivation with a non-zero reverse derivation d on N, and $a \in N$, then

(i) If $af(U) = 0$, then $a = 0$.

(ii) If $f(U)a = 0$,then $a = 0$.

Proof:

(i) We suppose that $af(U) = \{ 0 \}$, for $a \in N$. Then for all $r \in N$ and for all $b \in U$

$0 = a f(br)$

$= a (f(r)b + rd(b))$

$= af(r)b + ard(b)$

$= ard(b)$

Hence $aNd(b) = \{ 0 \}$, for all $b \in U$. Since N is prime near ring, then either $a = 0$ or $d(b) = 0$ but d be non-zero reverse derivation on N, then we get $a = 0$.

(ii) We suppose that $f(U)a = 0$, for all $a \in N$. Then for all $b, r \in U$,

$0 = f(rb)a$

$= (d(b)r + bf(r))a$

$= d(b)ra + bf(r)a$

$= d(b)ra$

Hence $d(b)Na = \{0\}$, for all $b \in U$. Since N is prime near ring, then either $a = 0$ or $d(b) = 0$, but d is a non-zero derivation on N, then we get $a = 0$.

Lemma 5.2.8 Let N be a prime near ring and U be a non-zero semigroup ideal of N. If d is a non-zero reverse derivation on N and $a \in U$ such that $ad(x)= 0$ (or) $d(x)a = 0$), for all $x \in U$. Then $a = 0$

Proof: We suppose that $ad(x) = 0$, for all $x \in U$.

By replacing x by yx, we have

$\Rightarrow$ $ad(yx) = 0$

$\Rightarrow$ $a(d(x)y + xd(y)) = 0$

$\Rightarrow$ $ad(x)y + axd(y) = 0$

⇨ $axd(y) = 0$,for all $x, y \in U$. Therefore $aNUd(y) = \{0\}$, since N is prime near ring we have either $a = 0$ (or) $Ud(y) = 0$ by lemma 4.3.5 $d(y) = 0$, for all $y \in N$

⇨ $d = 0$ on N which is contradiction. Thus $a = 0$

5.3. Commutativity of Prime Near Rings with Generalized Reverse Derivations in the Setting of Semigroup Ideals

In fact at section 4.4 at both theorem 4.4.5 and theorem 4.4.6.we have seen that certain results, a prime near ring with generalized derivation with non-zero derivation acting on semigroup ideals U become commutative ring. In this section we established some of similar results in the setting of semigroup ideals of a prime near ring N admitting a generalized reverse derivations then a prime near ring is commutative rings. We start by the following theorem.

Theorem 5.3.1 (Reddy et al., 2015) Let (f, d) be a generalized reverse derivation of N. If $f([x, y]) = 0$, for all $x, y \in N$, then N is a commutative ring.

Proof: We assume that $f([x, y] = 0$, for all $x, y \in N$.

By substituting xy instead of y, we obtain

⇨ $f([x, xy]) = 0$

⇨ $f(x[x, y]) = 0$

⇨ $f([x, y])x + [x, y]d(x) = 0$

Since the first term is zero, it is clear that

⇨ $xyd(x) = yxd(x)$ for all $x, y \in N$ (5.25)

We replace y by zy in equ. (5.25) and using this equation (5.25), we get

⇨ $[x, z]Nd(x) = 0$, for all $x, y \in N$.

Hence either $x \in Z(N)$ (or) $d(x) = 0$. Let $K = \{x \in N| x \in Z(N)\}$ and $L = \{x \in N|d(x) = 0\}$. Then K and L are two additive subgroups of $(N, +) = K \cup L$. However, a group Cannot be the union of proper subgroups, hence either $N = K$ (or) $N = L$.

Since $d \neq 0$, we conclude that N is a commutative ring.

Theorem 5.3.2 (Reddy *et al.,* 2015) Let (f, d) be a generalized reverse derivation of N, where d is non-zero. If $f(yx) - xy \in Z(N)$ (or) $f(yx) + xy \in Z(N)$, for all x, $y \in N$, then N is a commutative ring.

Proof: We assume $f(yx) - xy \in Z(N)$ which implies that $f(x)y + xd(y) - xy \in Z(N)$, for all x, $y \in N$ (5.26)

By replacing y by zy in equation (5.26), we get

⇨ $f(x)zy + xd(zy) - xzy \in Z(N)$, for all x, y, $z \in N$.

⇨ $[xyd(z) + z(f(x)y - xd(y) - xy), z] = 0$

⇨ $[xyd(z),z] = 0$, for all x, $y \in N$ (5.27)

Now, let us rewrite equation. (5.27) as the following

⇨ $x[y, z]d(z) + [x, z]yd(z) + xy[d(z),z] = 0$,for all x, y, $z \in N$ (5.28)

We replace x by wx in equation (5.28) and using equation (5.28), we get

⇨ $[w, z]xyd(z) = 0$, for all x, y, z, $w \in N$.

By putting $xy = r$, $r \in N$ then the above equation reduces to,

⇨ $[w, z]rd(z) = 0$

⇨ $[w, z]Nd(z) = 0$ By primeness of N, we have either $d(z) = 0$ (or) $[w, z] = 0$

Since by hypothesis d is non-zero then $[w, z] = 0$

Hence N is a commutative ring.

Similarly, the case $f(yx) + xy \in Z(N)$ implies the commutativity of N is clear because we can replace f by $(-f)$ in the above case as required.

Theorem 5.3.3 (Reddy *et al*., 2015) Let (f, d) be a generalized reverse derivation of N, where d is non-zero. $f(xy) - xy \in Z(N)$ (or) $f(xy) + xy \in Z(N)$, for all x, $y \in N$, then N is a commutative ring.

Proof: We have $f(xy) - xy \in Z(N)$, which implies that,

⇨ $[f(y)x + yd(x) - xy, r] = 0$, for all x, $y \in N$ and $r \in N$ (5.29)

Now equation.(5.29) can be rewritten as the following.

⇨ $f(y)[x, r] + [f(y),r]x + y[d(x),r] + [y, r]d(x)$

$= x[y, r] + [x, r]y$, for all $x, y \in N$ and $r \in N$ (5.30)

We replace x by xr in equation (5.30), we get,

⇨ $(f(y)[x, r] + [f(y), r]x + y[d(x), r] + [y, r]d(x))r + [x, r]d(r) + yx[d(r),r] + [y, r]xd(r) = xr[y, r] + [x, r]ry$, for all $x, y \in N$ and $r \in N$ (5.31)

From equations (5.30) and (5.31), we have

⇨ $[x, r][y, r] + x[[y, r], r] + [y, r]xd(r) + y[x, r]d(r) + yx[d(r),r] = 0$,

for all $x, y \in N$ and $r \in N$ (5.32)

We replace x by yx in equation (5.32) and using equation (5.32),

we get, ⇨ $[y, r]xyd(r) + [y, r]x[y, r] = 0$, for all $x, y \in N$ and $r \in N$ (5.33)

If we replace r by r + y in equation (5.33) and using (5.33), we get

⇨ $[y, r]xyd(y) = 0$,for all $x, y \in N$ and $r \in N$ (5.34)

Put $xy = z$ in equation (5.34), then we get,

⇨ $[y, r]zd(y) = 0$,for all $x, y, z \in N$ and $r \in N$ (5.35)

By primeness of N, either $d(y) = 0$ (or) $[y, r] = 0$

By hypothesis we have, d is non-zero. Therefore we have $[y, r] = 0$.

Hence N is a commutative ring.

Motivated by above Reddy *et al.* (2015) result we extend these results for semigroup ideals as follows

Theorem 5.3.4 Let (f, d) be a generalized reverse derivation on N and U be a non-zero semigroup ideal of N. If $f([x, y]) = 0$, for all $x, y \in U$, then N is a commutative ring.

Proof: We assume that $f([x, y]) = 0$, for all $x, y \in U$.

By substituting xy instead of y,

We obtain, $f([x, xy]) = 0$

⇨ $f(x[x, y]) = 0$

⇨ $f([x, y])x + [x, y]d(x) = 0$

Since the first term is zero, it is clear that

$xyd(x) = yxd(x)$ for all $x, y \in U$ (5.36)

We replace y by zy in equation (5.36) and using this equation (5.36), we get

$[x, z]Nd(x) = 0$, for all $x, y \in U$

Hence either $x \in Z(N)$ or $d(x) = 0$

Let $k = \{x \in N | x \in Z(N)\}$ and $L = \{x \in N | d(x) = 0\}$

Then k and L are two additive subgroups of $(N, +) = k \cup L$. However, a group cannot be the union of proper subgroups, hence either $N = k$ or $N = L$.

Since $d \neq 0$

We conclude that N is a commutative ring.

Theorem 5.3.5 Let (f, d) be a generalized reverse derivation on N. Where d is non-zero on N and U be a non-zero semigroup ideal of N. If $f(yx) - xy \in Z(N)$ for all $x, y \in U$, then N is a commutative ring.

Proof: We assume $f(yx) - xy \in Z(N)$ which implies that

$f(x)y + xd(y) - xy \in Z(N)$ for all $x, y \in U$ (5.37)

by replacing y by zy in equation (5.37), we get

$f(x)zy + xd(zy) - xzy \in Z(N)$ for all $x, y, z \in U$

⇨ $f(x)zy + x(d(y)z + yd(z)) - xyz$ for all $x, y, z \in U$

⇨ $f(x)zy + xyd(z) + xzd(y) - xyz$ for all $x, y, z \in U$

$\Rightarrow [xyd(z) + z(f(x)y - xd(y) - xy), z] = 0$

$\Rightarrow [xy\,d(z), z] = 0$ for all $x, y \in U$ (5.38)

Now, let us rewrite equation (5.38) as the following

$x[y, z]d(z) + [x, z]yd(z) + xy[d(z), z] = 0$, for all $x, y, z \in U$ (5.39)

We replace x by wx in equation (5.39) and using equation (5.39), we get

$[w, z]xy\,d(z) = 0$, for all $x, y, z, w \in U$

By putting $xy = r, r \in N$ then the above equation reduces to,

$[w, z]r\,d(z) = 0$

$\Rightarrow [w, z]N\,d(z) = 0$

By primness of N, we have either $d(z) = 0$ or $[w, z] = 0$

Since by the hypothesis d is non – zero then $[w, z] = 0$

Hence N is a commutative ring.

Theorem 5.3.6 Let (f, d) be generalized reverse derivation acting on N. Where d is non-zero on N and U be a non-zero semigroup ideal of N. If $f(yx) + xy \in Z(N)$, for all $x, y \in U$, then N is commutative ring.

Proof: We suppose that $f(yx) + xy \in Z(N)$, for all $x, y \in U$

$\Rightarrow f(x)y + xd(y) + xy$, for all $x, y \in U$

By replacing f by $(-f)$, we get

$xd(y) - f(x)y + xy$, for all $x, y \in U$ (5.40)

By replacing x by xz in equation (5.40), we get

$xzd(y) - f(xz)y + xzy$, for all $x, y, z \in U$

⇨ $xzd(y) - (f(z)x \; - zd(x))y \; + xzy$, for all x, y, z $\in$ U

⇨ $xzd(y) - f(z)xy + zyd(x) + xzy$, for all x, y, z $\in$ U

⇨ $[xzd(y) - \; y(f(z)x - zd(x) - xz), -y] = 0$, for all x, y, z $\in$ U

⇨ $[xzd(y), -y] \; = \; 0$, for all x, y, z $\in$ U (5.41)

Now, let as rewrite equation (5.41) as follows

$x[z, -y]d(y) + [z, -y]xd(y) + xz[d(y), -y] \; = \; 0$, for all x, y, z $\in$ U (5.42)

Now, replace x by wy in the equation (5.42), we get

$[z, -y]$ wy $d(y) = 0$, by putting wy $=$ r, where r $\in$ N.

⇨ $[z, -y]$ r $d(y) \; = \; 0$, for all x, y, z $\in$ U

⇨ $[z, -y]$ N $d(y) = 0$, for all x, y, z $\in$ U,

by primeness of N, either of $d(y) = 0$ or $[z, -y] = 0$.

but by the hypothesis d $\neq$ 0, then $[z, -y] \; = \; 0$.

Thus N is commutative.

Theorem 5.3.7 Let (f, d) be generalized reverse derivation acting on N, where d is non – zero on N and U be a non- zero semigroup ideal of N. If f(xy) – xy $\in$ Z(N), for all x, y $\in$ U, then N is a commutative ring.

Proof: We have f (xy) – xy $\in$ Z(N), which implies that

⇨ $[f(y)x + yd(x) - xy, r] \; = \; 0$, for all x, y $\in$ U and r $\in$ N (5.43)

Now equation (5.43) can be written as the following.

⇨ $f(y)[x, r] + [f(y), r]x + y[d(x), r] + [y, r]d(x) \; = \; x[y, r] + [x, r]y$

for all x, y $\in$ U And r $\in$ N (5.44)

We replace x by xr in equation (5.44) we get

$(f(y)[x, r] + [f(y), r]x + y[d(x), r] + [y, r]d(x))r$

$+[x, r]d(r) + yx[d(r), r] + [y, r]x\,d(r) = xr[y, r]$

$+[x, r]ry$, for all $x, y \in U$ and $r \in N$ (5.45)

From equations (5.44) and (5.45) we have

$[x, r][y, r] + x[[y, r], r] + [y, r]x\,d(r) + y[x, r]d(r) + yx[d(r), r] = 0$

for all $x, y \in U$ and $r \in N$ (5.46)

We replace x by yx in equation (5.46) and using equation (5.46) we get

$[y, r]xy\,d(r) + [y, r]x[y, r] = 0$ for all $x, y \in U$ and $r \in N$ (5.47)

If we replace r by r + y in equation (5.47) and using equation (5.34) we get

$[y, r]xy\,d(y) = 0$ for all $x, y \in U$ and $r \in N$ (5.48)

Put $xy = z$ in equation 5.48, then we get

$[y, r]z\,d(y) = 0$ for all $x, y, z \in U$ and $r \in N$ (5.49)

By primness of N, either $d(y) = 0$ or $[y, r] = 0$

By the hypothesis we have, d is non –zero. Therefore, we have $[y, r] = 0$.

Thus, N is a commutative ring.

5.4. Generalized Reverse Derivations on Prime near Rings acting as a Homomorphism or Anti-Homomorphism in the Setting of Semigroup Ideals

Definition 5.4.1 (Satyanarayana and Prasad, 2013) Let N and N_1 be near rings. A mapping h: N → N_1 is defined a homomorphism or near ring homomorphism) if

$h(m + n) = h(m) + h(n)$ and $h(mn) = h(m) \cdot h(n)$, $\forall m, n \in N$.

Definition 5.4.2 let N and N_1 be near rings. A mapping f: N → N_1 is defined a anti-homomorphism or near ring anti-homomorphism if $f(x + y) = f(x) + f(y)$ and $f(xy) = f(y) f(x)$ $\forall$ x, y ∈ N.

Theorem 5.4.1 (Reddy *et al.,* 2015) Let (f, d) be a non-zero generalized reverse derivation of N. If f acts as a homomorphism on N, then f is the identity map.

Proof: We assume that f acts as a homomorphism on N.

Then, $f(yx) = f(y) f(x)$

$$= d(x)y + xf(y) \text{ for all } x, y \in N \text{ by definition} \quad (5.50)$$

We replace y by yx in equation (5.50), we get

$$f(yx)f(x) = d(x)yx + xf(yx)$$

$$\Rightarrow (d(x)y + xf(y))f(x) = d(x)yx + xf(y)f(x)$$

By Lemma:4(ii), we have

$$d(x)yf(x) + xf(y)f(x) = d(x)yx + xf(y)f(x)$$

$$\Rightarrow d(x)yf(x) - d(x)yx = 0$$

$$\Rightarrow d(x)y(f(x) - x) = 0, \text{ for all } x, y \in N.$$

$$\Rightarrow d(x)N(f(x) - x) = 0, \text{ for all } x, y \in N.$$

Since N is prime, we have either $d = 0$ (or) $(f(x) - x) = 0$.

But by hypothesis, we have, d is non-zero

$$\Rightarrow d \neq 0. \text{ Therefore } (f(x) - x) = 0$$

$$\Rightarrow f(x) = x, \text{ for all } x \in N. \text{ Thus, f is the identity map.}$$

Theorem 5.4.2 (Reddy *et al.,* 2015) Let (f, d) be a non-zero generalized reverse derivation of N. If f acts as an anti-homomorphism on N, then f is the identity map.

Proof: By the hypothesis, we have, $f(yx) = f(x)f(y)$

$$= d(x)y + xf(y), \text{ for all } x, y \in N \quad (5.51)$$

We replace y by zy in equation (5.51), we obtain

$$f(x)f(zy) = d(x)zy + xf(zy)$$

$$\Rightarrow \quad f(x)f(y)f(z) = d(x)zy + xf(y)f(z)$$

$$\Rightarrow \quad f(yx)f(z) = d(x)zy + xf(y)f(z)$$

$$\Rightarrow \quad (d(x)y + xf(y))f(z) = d(x)zy + xf(y)f(z)$$

By Lemma 4 (ii), we have,

$$d(x)yf(z) + xf(y)f(z) = d(x)zy + xf(y)f(z)$$

$$\Rightarrow \quad d(x)yf(z) - d(x)yz = 0$$

$$\Rightarrow \quad d(x)y(f(z) - z) = 0, \text{ for all } x, y, z \in N.$$

$$\Rightarrow \quad d(x)N(f(z) - z) = 0, \text{ for all } x, y, z \in N.$$

Since N is prime, we have, either $d = 0$ (or) $(f(z) - z) = 0$.

But, by hypothesis, d is non-zero. Therefore $(f(z) - z) = 0$.

$\Rightarrow$ $f(z) = z$, for all $z \in N$. Thus, f is the identity map.

Motivated by above results we extend these results for semigroup ideals

Theorem 5.4.3 Let (f, d) be a non-zero generalized reverse derivation on N and U be a non-zero semigroup ideal of N. If f acts as a homomorphism on U, then f is an identity map on U.

Proof: we assume that f acts as a homomorphism on U.

Then $f(yx) = f(y) f(x) = d(x)y + xf(y)$, for all $x, y \in U$ (5.52)

We replace y by yx in equation 5.52, we get

$f(yx)\,f(x) = d(x)yx + xf(yx)$

⇨ $(d(x)y + xf(y))f(x) = d(x)yx + xf(y)f(x)$

⇨ $d(x)yf(x) + xf(y)f(x) = d(x)yx + xf(y)f(x)$

⇨ $d(x)yf(x) - d(x)yx = 0$

⇨ $d(x)y(f(x) - x) = 0$, for all $x, y \in U$

⇨ $d(x)N(f(x) - x) = 0$, for all $x, y \in U$

Since N is prime, we have either $d = 0$ or $(f(x) - x) = 0$

But by hypothesis, we have, d is non – zero

⇨ $d \neq 0$ Therefore $(f(x) - x) = 0$

⇨ $f(x) = x$, for all $x \in U$

Thus f is the identity map.

Theorem 5.4.4 Let (f, d) be a non –zero generalized reverse derivation on N and U be a non-zero semigroup ideal of N. If f acts as an anti – homomorphism on U, then f is the identity map on U.

Proof: By the hypothesis, we have $f(yx) = f(x)f(y)$

$= d(x)y + xf\,y)$ for all $x, y \in U$ (5.53)

We replace y by zy in equation 5.53, we obtain

$f(x)f(zy) = d(x)zy + xf(zy)$

⇨ $f(x)f(y)f(z) = d(x)zy + xf(y)f(z)$

⇨ $f(yx)f(z) = d(x)zy + xf(y)f(z)$

⇨ $(d(x)y + xf(y))f(z) = d(x)zy + xf(y)f(z)$

⇨ $d(x)yf(z) + x\,f(y)f(z) = d(x)zy + xf(y)f(z)$

⇨ $d(x)yf(z) - d(x)yz = 0$

⇨ $d(x)y(f(z) - z) = 0$, for all $x, y, z \in U$

⇨ $d(x)N(f(z) - z) = 0$, for all $x, y, z \in U$

Since N is prime, we have either $d = 0$ or $f(z) – z = 0$

But by the hypothesis, d is non – zero. Therefore $f(z) - z = 0$

⇨ $f(z) = z$, for all $z \in U$.

Thus, f is the identity map.

5.5. Product of Generalized Reverse Derivations on Prime Near Rings with Semigroup Ideals

(Posner, 1957) established a very striking result which states that if R is a prime ring of characteristic different from two and d_1, d_2 are derivations of R such that the iterate d_1d_2 is also a derivation, then at least one of d_1 or d_2 is zero.

(Wang, 1994) obtained in near rings which is as follows

Let N be a 2-torsion free prime ring and d_1, d_2 be derivations of N such that d_1d_2 is also a derivation. Then $[d_1(x), d_2(x)] = 0$ for all $x, y \in N$ if and only if either $d_1 = 0$ or $d_2 = 0$.

Theorem 5.5.1 (Golbasi, 2006) Let (f, d) and (g, h) be two generalized derivations of N. If N is 2-torsion free and $f(x)h(y) = -g(x)d(y)$ for all $x, y \in N$, then $f = 0$ or $g = 0$.

Proof: If $h = 0$ or $d = 0$, then the proof of the theorem is obvious. So, we may assume that $h \neq 0$ and $d \neq 0$. Therefore we know that $(N, +)$ is abelian by lemma 4.4.2(ii) Now suppose that

$$f(x)h(y) + g(x)d(y) = 0 \text{ for all } x, y \in N. \tag{5.54}$$

Replacing x by uv in this equation (5.54) and using the hypothesis, we get

$$f(uv)h(y) + g(uv)d(y) = uf(v)h(y) + d(u)vh(y) + ug(v)d(y) + h(u)vd(y) = 0,$$

and so

$d(u)vh(y) = -h(u)vd(y)$ for all $u, v, y \in N$. (5.55)

Taking yt instead of y in the above relation, we obtain

$d(u)vh(y)t + d(u)vyh(t) = -h(u)vd(y)t - h(u)vyd(t)$.

That is,

$d(u)vyh(t) = -h(u)vyd(t)$ for all $u, v, y, t \in N$. (5.56)

Replacing y by h(y) in (5.56) and using this relation, we have

$h(u)N(d(y)h(t) - h(y)d(t)) = 0$ for all $u, y, t \in N$.

Since N is a prime near-ring and $h \neq 0$, we obtain that

$d(y)h(t) = h(y)d(t)$ for all $y, t \in N$. (5.57)

Now again taking uv instead of x in the initial hypothesis, we get

$f(u)vh(y) + ud(v)h(y) + g(u)vd(y) + uh(v)d(y) = 0$.

Using (5.57) yields that

$f(u)vh(y) + 2uh(v)d(y) + g(u)vd(y) = 0$ for all $u, v, y \in N$.

Taking h(v) instead of v in this equation, we arrive at

$f(u)h(v)h(y) + 2uh^2(v)d(y) + g(u)h(v)d(y) = 0$.

By the hypothesis and (5.57), we have $0 = -g(u)d(v)h(y) + 2uh^2(v)d(y) + g(u)h(v)d(y)$

$$= -g(u)h(v)d(y) + 2uh^2(v)d(y) + g(u)h(v)d(y),$$

and so

$2uh^2(v)d(y) = 0$ for all $u, v, y \in N$. Since N is a 2-torsion free prime near-ring, we obtain that $h^2(N)d(N) = 0$. By applying lemma 4. 2.2 (ii) gives that $h = 0$.

This contradicts by our assumption.

Theorem 5.5.2 (Golbasi, 2006) Let (f, d) and (g, h) be two generalized derivations of N. If (fg, dh) acts as a generalized derivation on N, then f = 0 or g = 0.

Theorem 5.5.3 (Reddy *et al.,* 2015) Let (f, d) and (g, h) be two generalized reverse derivations of N. If (fg, dh) acts as a generalized reverse derivation on N, then (N,+) is abelian.

Proof: By calculating fg(xy) in two different ways, then we get

$f(y)h(x) + g(y)d(x) = 0$, for all $x, y \in N$ We replace x by x + z, we get

$f(y)h(x) + f(y)h(z) + g(y)d(x) + g(y)d(z) = 0$

By using the hypothesis, we get

$f(y)h(x, z) = 0$, for all $x, y, z \in N$. By Lemma 5.2.3(ii), we have

$h(x, z) = 0$, for all $x, z \in N$. For any $w \in N$, we have

$h(wx, wz) = 0$

$\Rightarrow h(w(x, z)) = 0$

$\Rightarrow h(x, z)w + (x, z)h(w) = 0$

$\Rightarrow (x, z)h(w) = 0$, for all $w, x, z \in N$. By Lemma 4.4.2(ii) implies that (N, +) is abelian.

Theorem 5.5.4 (Reddy *et al.,* 2015) Let N be a near ring of char≠2, (f, d) and (g, h) be two generalized reverse derivations on N. If (fg, dh) acts as a generalized reverse derivation on N,

then f = 0 (or) g = 0.

Proof: If h = 0 (or) d = 0, then the proof of the theorem is obvious. So, we may

Assume that h ≠ 0 and d ≠ 0. Therefore, we know that (N, +) is abelian by Theorem 5.5.3

By calculating fg(xy)in two different ways, then we get

$$f(y)h(x) + g(y)d(x) = 0, \text{for all } x, y \in N \quad (5.58)$$

We replace y by uv in the equation (5.58) and using equation (5.58), we get

$f(uv)h(x) + g(uv)d(x) = 0$

$\Rightarrow d(v)uh(x) + vf(u)h(x) + h(v)ud(x) + vg(u)d(x) = 0$

$$\Rightarrow d(v)uh(x) = -h(v)ud(x), \text{ for all } u, v, x \in N \quad (5.59)$$

By taking tx instead of x in equation (5.59), we get,

$\Rightarrow \quad d(v)uh(x)t + d(v)uxh(t) = -h(v)ud(x)t - h(v)uxd(t)$

$\Rightarrow \quad d(v)uxh(t) = -h(v)uxd(t)$, for all u, v, x, t $\in$ N (5.60)

We replace x by h(x) and using equation (5.58), we get,

$h(v)N(d(x)h(t) - h(x)d(t)) = 0$,for all v, x, t $\in$ N.

Since N is a prime near ring and h $\neq$ 0.

We obtain that,

$d(x)h(t) = h(x)d(t)$, for all x, t $\in$ N (5.61)

Now again taking uv instead of y in equation (5.58), we get

$f(v)uh(x) + vd(u)h(x) + g(v)ud(x) + vh(u)d(x) = 0$

$\Rightarrow \quad f(v)uh(x) + g(v)ud(x) + 2vh(u)d(x) = 0$,for all u, v, x $\in$ N

We take h(u) Instead of u in the above equation, then we get,

$f(v)h(u)h(x) + g(v)h(u)d(x) + 2vh(h(u))d(x) = 0$ By equation (5.58), we get,

$-g(v)d(u)h(x) + g(v)h(u)d(x) + 2vh^2(u)d(x) = 0$ By equation (5.61)

$\Rightarrow \quad -g(v)h(u)d(x) + g(v)h(u)d(x) + 2vh^2(u)d(x) = 0$

$\Rightarrow \quad 2vh^2(u)d(x) = 0$, for all u, v, x $\in$ N. Since N is a prime near ring of char $\neq$2.

We obtain that

$h^2(u)d(x) = 0$

$\Rightarrow \quad h^2(N)d(N) = 0$. By Lemma 1(iii) and (iv) we have, h = 0.

This contradicts our assumption.

Initiated by above observations we proved the following results for generalized reverse derivation with semigroup ideals

Theorem 5.5.5 Let (f, d) and (g, h) be two generalized reverse derivations on N and U be a non-zero semigroup ideal of N.. If (fg, dh) acts as a generalized reverse derivation on U with associated reverse derivation d and h on U, then (N, +) is abelian.

Proof: By calculating fg(xy) in two different ways, then we get

$f(y)h(x) + g(y)d(x) = 0$, for all $x, y \in U$

We replace x by x + z, we get

$f(y)h(x) + f(y)h(z) + g(y)d(x) + g(y)d(z) = 0$

By using the hypothesis on lemma 5.5.2 we get

$f(y)h(x, z) = 0$ for all $x, y, z \in U$

By lemma 5.1.2 (a), we have $h(x, z) = 0$ for all $x, z \in U$.

For any $w \in U$, we have,

$h(wx, wz) = 0$

⇨ $h(w(x, z)) = 0$

⇨ $h(x, z)w + (x, z)h(w) = 0$

⇨ $(x, z)h(w) = 0$, for all $w, x, z \in U$.

By lemma 5.1.2 (b) implies that (N, +) is abelian.

Theorem 5.5.6 Let N be a 2-torsion free prime near ring, (f, d) and (g, h) be two generalized reverse derivations on N and U be a non-zero semigroup ideal of N. If (fg, dh) acts as a generalized reverse derivation on U with associated reverse derivation on U, then f = 0 (or) g = 0 on U.

Proof: We assume that $h \neq 0$ and $d \neq 0$.

Therefore, we know that (N, +) is abelian by Theorem 5.5.5

By calculating fg(xy)in two different ways, then we get

$f(y)h(x) + g(y)d(x) = 0$, for all $x, y \in U$ (5.62)

We replace y by uv in the equ.(5.62) and using equ.(5.62), we get

$$f(uv)h(x) + g(uv)d(x) = 0$$

$\Rightarrow$ $d(v)uh(x) + vf(u)h(x) + h(v)ud(x) + vg(u)d(x) = 0$

$\Rightarrow$ $d(v)uh(x) = -h(v)ud(x)$, for all u, v, x $\in U$ by lemma 5.2.7 (5.63)

By taking tx instead of x in equ. (5.63), we get

$$d(v)uh(x)t + d(v)uxh(t) = -h(v)ud(x)t - h(v)uxd(t)$$

$\Rightarrow$ $d(v)uxh(t) = -h(v)uxd(t)$, for all u, v, x, t $\in$ U by lemma 5.1.1.2 (5.64)

We replace x by h(x) and using equ.(5.63), we get

$$h(v)U(d(x)h(t) - h(x)d(t)) = 0, \text{ for all } v, x, t \in U.$$

Since N is a prime near ring and U is semigroup ideal

and also $h \neq 0$,we obtain that

$d(x)h(t) = h(x)d(t)$, for all x, t $\in$ U (5.65)

Now

Again taking uv instead of y in equ (5.62),

We get,

$$f(v)uh(x) + vd(u)h(x) + g(v)ud(x) + vh(u)d(x) = 0$$

$\Rightarrow$ $f(v)uh(x) + g(v)ud(x) + 2vh(u)d(x) = 0$, for all u, v, x $\in$ U.

We take h(u) instead of u in the above equation,

then we get,

$$f(v)h(u)h(x) + g(v)h(u)d(x) + 2vh(h(u))d(x) = 0$$

By equ. (5.65), we get

$g(v)d(u)h(x) + g(v)h(u)d(x) + 2vh^2(u)d(x) = 0$

By equ.(5.62), we get

$- g(v)h(u)d(x) + g(v)h(u)d(x) + 2vh^2(u)d(x) = 0$

⇨ $2vh^2(u)d(x) = 0$, for all $u, v, x \in U$

Since N is a prime near ring of two- torsion free we obtain that

$h^2(u)d(x) = 0$

⇨ $h^2(N)d(U) = 0$. By Lemma 5.2.2 (a) and (b)

We have $h = 0$.

This contradicts our assumption.

Thus $f = 0$ or $g = 0$

6. SUMMARY AND CONCLUSION

6.1. Summary

This thesis presents some results on generalized reverse derivations on prime near rings with semigroup ideals. We have first discussed the general introduction, which includes the historical background of the study, statement of the problem and objectives of the study. Then literature reviews of generalized reverse derivations on prime near rings were presented. Additionally we have discussed the preliminary concepts and definitions and some results in prime near rings to make the concepts to be clear. Some results of reverse derivations on near rings contributions to prime near rings involving generalized reverse derivations are presented. Generalized reverse derivations on prime near rings acts as homomorphism or anti-homomorphism on semigroup ideals of N were an identity map was also discussed. In this thesis also we have discussed some results of extensions on generalized reverse derivations on prime near rings with semigroup ideals that shows prime near ring is commutative. Finally, we have showed that if product of two generalized reverse derivations acting as generalized reverse derivations on semigroup ideal of N, then at least one of generalized reverse derivation is zero.

6.2. Conclusion

In this thesis, to discuss about generalized reverse derivations on prime near rings we have used near rings with associated non-zero reverse derivations and we can conclude that primeness near rings is very important tool for near ring is abelian. A generalized reverse derivations on prime near rings with semigroup ideals of (f, d) which acts as a homomorphism or anti-homomorphism on N is identity map. Also we have showed that near rings is commutative ring with generalized reverse derivations of prime near rings with semigroup ideals.

6.3. Recommendation

This thesis done on some results on generalized reverse derivations on prime near rings with semigroup ideals. For further study researchers can do on multiplicative (generalized) reverse derivations on prime near rings with semigroup ideals.

7. REFERENCES

Aboubakr, A. and Gonzalez, S. 2015. Generalized reverse derivations on semi prime rings. *Siberian Mathematical Journal, 56(2): 199-205.*

Ali *et al. (*2013). Generalized derivations on prime near rings. International *Journal of Mathematics and Mathematical Sciences, ID 170749.*

Ali *et al. (*2013). Generalized derivations on prime near rings II. *African Mathematical Union and Springer-Verlag Berlin Heidelberg.*

Ali and Bano (2018). Multiplicative (generalized) reverse derivations on semiprime ring. *Department of Mathematics, Aligarh Muslim University, Aligarh, India, 11(3):717-729.*

Argac, N. 2004. On near-rings with two sided α-derivations. *Turkey Journal of Mathematics 28:195-204.*

Ashraf, M. and Ali, S. 2008. On (α, τ)-derivations of prime near-rings II. *Sarajevo Journal of Mathematics, 4(16):1-8.*

Ashraf *et al*. 2005. On generalized derivations of prime-rings*. Southeast Asian Bulletin of Mathematics, 29(4):669–675.*

Ashraf *et al*. 2014. "On semigroup ideals and n- derivations in near-rings". *Science Direct Journal of Taibah University for Science, 9:126 – 132.*

Ashraf, M. and Siddeeque, M. 2014. "On generalized n – derivations in near- rings". *Palestine Journal of Mathematics, 3(1):468-480.*

Ashraf, M. and Siddeeque, M. 2015. Generalized derivations on semigroup ideals and commutativity of prime near-rings*. Rend. Sem. Mat. Univ. Pol. Torino, 73(2): 217 – 225.*

Asma, A. and Inzamam, U. H. 2020. Commutativity of a 3-prime near ring satisfying certain differential identities on jordan ideals*. Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India, 1: 1-11.*

Awtar, R. 1973. Lie ideals and Jordan structures in prime rings with derivations. *Proceedings American Mathematical Society, 41:.67-74.*

Beidar *et al.* 1996. Rings with Generalized Identities. *Pure and Applied Mathematics, Marcel Dekker, New York.*

Beidleman, J. C. 1967. Strictly prime d — g near rings. *Math. Zaitech, 100: 97-105.*

Bell, H. E. 1980. On Commutativity of periodic rings and near rings. *Acta Mathematical Academical Scientiarum Hungarica, 36: 293–302.*

Bell, H. E. 1997. On Derivations in Near Rings II. Proceedings of the Conference on Near rings & Near fields. *Hamburg, Germany, 191-197.*

Bell, H. E. 2008. On prime near rings with generalized derivations*, Department of Mathematics, Faculty of Mathematics and Science, Brock University, St. Catharines, Ontario, Canada.*

Bell, H. E. 2008. On prime near-rings with generalized derivations. *International Journal of Mathematics and Mathematical Sciences, ID 490316.*

Bell, H. E. and Daif, M. N. 1994. On commutativity and strong commutativity preserving maps. *Canada Mathematical Bulliten,.37: 443-447.*

Bell, H. E. and Daif, M. N. 1995. On Derivations and Commutativity in Prime Rings. *Acta Mathematical Hungary, 66 (4): 337-343.*

Bell, H. E. and Kappe, L. C. 1989. Rings in which derivations satisfying certain algebraic condition. *Acta Mathematical Hungary, 53(34): 339-346.*

Bell, H. E. and Mason, G. 1987. On derivations in near rings, Near rings and Near fields. *North-Holland Mathematical Studies, 137: 31-35.*

Bell, H.E. and Mason, G. 1992. On derivations in near-rings and rings, *Mathematics Journal of Okayama University, 34:135-144.*

Bharathi, M. and Jayalakshmi, K. 2017. "(1,α)-Derivations on lie Ideals in σ- prime near-rings". *Mathematical sciences International Research journal, 6(1): 22-25.*

Booth, G. L. and Groenewald, N. J. 1998. On strongly prime near rings. *Indian Journal of Mathematics, 40(2): 113 – 121.*

Boua Abdelkarim. 2012. Some conditions under which prime near-rings are commutative rings. *International Journal of Open Problems Computational Mathematics 5(2).*

Boua *et al*. 2016. Some results on 3-prime near-rings with derivations. *Indian Journal Pure Applied Mathematics 47(4):705-716*.

Boua *et al.* 2016. "On 3-prime near-ring with generalized derivations". *Palestine Journal of Mathematics, 5(1): 12-16.*

Boua, A. and Oukhtite, L. 2011.Derivations on prime near-rings. *International Journal on Open Problems Computational Mathematics, 4(2):162-167.*

Brešar, M. 1991. On the distance of composition of two derivations to the generalized derivations*, Glasgow Mathematical Journal, 33:89 -93.*

Bresar, M. and Vukman, J. 1989. On some additive mappings in rings with involution. *Aequationes mathematical, 38: 178-185.*

Clay, J. R. 1968. The near rings on groups of lower order. *Mathematical Z, 104: 364-371.*

Deng*, et al*. (1998). On commutativity of near rings with derivations. *Mathematical Proceedings of the Royal Irish Academy. 98A (2): 217-222.*

Dickson, L.1905. "Definitions of a group and a field by independent postulates''. Trans. *Math. Sac. 6: 198-204.*

Golbasi, O. 2005. Some results on near-rings with generalized derivation. Commun. *Fac.Sci. Ank. Ser, 51(2): 21–26.*

Golbasi, O. 2006. Notes on prime near-rings with generalized derivation. *Southeast Asian Bulletin of Mathematics, 30: 49 -54.*

Golbasi, O. 2006. On generalized derivations of prime near-rings. *Hacettepe Journal of Mathematics and Statistics, 35(2): 173−180.*

Groenewald, N. 2003. “Semi-uniformly strongly prime near-rings”. *Indian Journal Mathematics, 45 (3): 241–250.*

Herstein, I. N. 1957. Jordan derivations of prime rings. *Proceedings of the American Mathematical Society, 8(6): 1104 - 1110.*

Herstein, I. N. 1976. Rings with Involution. Chicago Lectures in Mathematics. *The University Chicago press. Chicago.*

Hvala, B. 1998. Generalized derivations in rings. *Comm. Algebra, 26: 1147-1166.*

Ibraheem, A. M. 2018. “The commutativity of prime near- rings”. *International Journal of Research-Granthaalayah, 6(2): 339-345.*

Jamal, M. R. 2008. A study of derivations in near-rings. M. Phil Thesis. Aligarh Muslim University Aligarh, India.

Jun *et al.* (2001). “Fuzzy maximal ideals of Gamma near-rings”, *Turkish, Journal of Mathematics, 25: 457-463.*

Khan, M. R. and Hasnain, M. M. 2013. On Generalized Derivation On Prime Near Ring. *Chinese Journal of Mathematics Hindawi Publishing Corporation.*

Kamal. A. M. and Al-Shaalan K. H. 2013. Existence of derivations on near rings. *Mathematical Institute Slovak Academy of Sciences, 63(3): 431–448.*

Kamal. A. M. and Al-Shaalan K. H. 2013. Commutativity of rings and near-rings with generalized derivations. *Indian Journal of Pure Applied Mathematics, 44(4): 473-496*

Kon, B. Ch. 1988. Prime Near rings. *Community of Korean Mathematical Society 2: 125-131.*

Ligh, S. 1969. "On Boolean near rings". *Bulletin Austral mathematical society, 1: 375-379.*

Ligh, S. and Luh, J. 1976. Some commutativity theorems for rings and near rings. *Acta Mathematical Academies Scientiarum Hungarica, 28(1-2): 19-23.*

Majeed, H. A. and Ahmed, H. A. 2011. Semigroup ideal in Prime Near Rings with Derivations. *Baghdad Science Journal, 8(3).*

Narayana *et al*. (2020). An innovative study on reverse derivation in near rings, *European Journal of Molecular & Clinical Medicine. 07(03):2020.*

Nowicki, A. 1987. Integral derivations. *Journal of Algebra, 110 (1): 262-276.*

Pilz, G. 1983. Near-Rings, 2nd edition. *North – Holland Publishing Company Amsterdam-New York-Oxford.*

Posner, E. C. 1957. Derivations in prime rings. *Proceeding American Mathematical Society, 8:1093-1100.*

Reddy *et al*. (2015). Generalized Reverse derivations on Prime Near Rings. *International Journal of Research in Mathematics & Computation, 3(2): 01-09.*

Rehman, N. 2002. On Commutativity of Rings with Generalized Derivations. *Proceeding American Mathematical Society, 41: 67-74.*

Saed, A. I. 2018. "On Semigroup Ideals and Left Generalized (θ, θ) $-$4-Derivations in Prime Near Rings ". *International Journal of Mathematics Trends and Technology, 2(57).*

Sammana *et al.* (2015). A study of near-rings with generalized derivations. *Journal of Taibah University for Science, 9:407–413.*

Satyanarayana, B. and Prasad, K. 2013. 'Near rings, Fuzzy Ideals and Graph Theory', *CRC Press (Taylor & Francis), Chapman & Hall (New York, London), 13: 9781439873106.*

Satyanarayna, B. B. 1984. "Contributions to near-rings". Doctoral Thesis, *Nagarjuna University.*

Suvarna, K. and Irfana, D. S. 2010. Generalized derivations on Prime Near Rings. *Indian Journal of Mathematics and Mathematical Sciences, 6: 9-13.*

Tiwari *et al.* (2018). Some theorems of commutativity on semiprime rings with mappings. *Southeast Asian Bulletin of Mathematics, 42: 579-592.*

Wang, K. 1994. Derivations in prime near rings. *Proceedings of American Mathematical Society, 121: 361-366.*